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Young supertableaux of the basic Lie superalgebras†

J P Hurni

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland

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Abstract. Among the simple Lie superalgebras (LSA), the basic classical ones are well known, together with many of their finite-dimensional representations. This is particularly true for the typical representations which are known to be completely reducible.

It is well known that reducible but indecomposable representations of the basic LSA are not completely characterised by their highest weight (HW) or the eigenvalues of their Casimir operators. In fact, there may be several different finite representations characterised by the same HW. Such representations are called non-typical or atypical.

The aim of this paper is to establish a complete characterisation of the typical and non-typical representations by means of properly defined Young supertableaux (YST). Therefore, the definition we are going to use for our YST differs slightly from the standard definitions. The examples given throughout this paper help the illustration of this new interpretation for the YST which makes possible the complete classification of the indecomposable representations characterised by a generalised notion of HW.

1. Introduction

Since 1978 many authors have attempted to present, in a less abstract manner, the general results obtained by Kac in his fundamental paper *Representations of Classical Lie Superalgebras* (Kac 1978).

Many approaches to the representations of these superalgebras were successful: by their explicit construction from the HW according to the above-mentioned paper (Farmer and Jarvis 1984, Hurni and Morel 1982, 1983), generalisation of Young tableaux to the LSA case (Abramsky and King 1970, Balantekin and Bars 1981, Bars *et al* 1983, King 1970, 1982), super Gel'fand-Zetlin bases (Chen and Chen 1983), the method of creation and annihilation operators (Dun-Sang Tang 1984), superfields (Farmer and Jarvis 1983) and so on (see, for example, Scheunert (1985) and references therein).

As physicists, several authors were by tradition essentially interested in the finite-dimensional irreducible representations (IR) of the simple Lie algebras (LA) (partially because of the complete reducibility theorem for the finite representations and partially because of their unitarisability in the compact case: the case of the gauge theories) and stuck to this attitude in the LSA case.

However, complete reducibility is not valid for all the simple LSA (in fact, the osp($1/N$) are the only exceptions for which complete reducibility does hold), and not even for the so-called basic ones which are the closest to the simple LA (the other simple LSA do not have Cartan matrices). Finite reducible but indecomposable representations may appear by simply making the tensor product of irreducible

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representations (IR). We sall them semi-reducible representations (SRR). They are usually perceived as pathologic and indirectly this paper aims to fight this idea. In fact, the SRR share many properties with some infinite-dimensional indecomposable representations of the LA.

In contrast to the irreducible case, there is no complete classification of the finite SSR. Only the ones defined by their HW are known, as soon as we have recognised the many invariant subspaces of the maximal representation (MR) built from a given highest weight. It remains to divide this MR by each of its invaiiant subspaces to get all the inequivalent SRR and the IR having this same HW.

Another category of SRR is introduced in this paper: the representations characterised by a pseudo-highest weight (PHW). Note that in the setting of the LA, such PHW are needed to characterise the infinite unitarisable IR of the non-compact semi-simple Lie algebra.

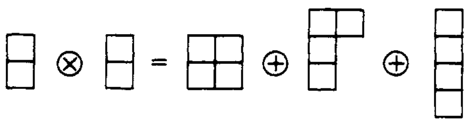
For su(N), Young tableaux (YT) have two possible meanings.

Firstly, they explain in which manner an IR may be obtained by the tensor product of the standard N-dimensional representation denoted □. For example □□ and  are the IR obtained by taking the symmetrised (respectively antisymmetrised) tensor product of the standard representation by itself.

If we denote the basis vectors of □ by the N covariant vectors t^i, $i = 1, \ldots, N$, then the $\frac{1}{2}N(N+1)$ ones of □□ and the $\frac{1}{2}N(N-1)$ ones of  are denoted by $T^{(ij)} \equiv (t^i t^j + t^j t^i)$ and $T^{[ij]} \equiv (t^i t^j - t^j t^i)$ respectively.

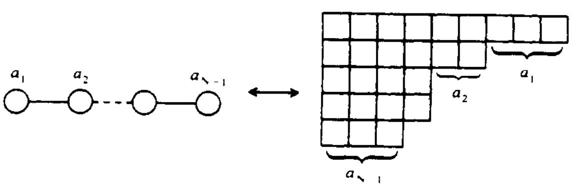
The contravariant vectors t_i, $i = 1, \ldots, N$, which describe the complex conjugate representation ⊡, are linearly related to the covariant ones by the totally antisymmetric, invariant, Levi-Civita ε tensor. Since ⊡ =  $N-1$, contravariant indices are superfluous.

Tensor products of arbitrary IR may be reduced using a simple algorithm: the Littlewood-Richardson rule (Hammermesh 1962, Littlewood 1950). For example

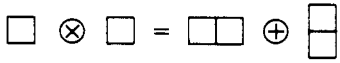


$\forall$su(N).

Secondly, YT can be used as a graphical notation in order to specify the non-negative integers a_i (given by the formula $a_i = 2(\Lambda, \alpha_i)/(\alpha_i, \alpha_i)$, where Λ is the HW and the α_i are the simple roots of su(N)) which characterise completely the finite-dimensional IR. These numbers, also called Dynkin labels or indices, are usually written as the components of the so-called 'highest weight vector' $\boldsymbol{a} = (a_1, \ldots, a_{N-1})$; however, in order to avoid confusion when considering other types of L(S)A, we will write them in a Dynkin diagram, above the nodes corresponding to the simple roots:



In the so(N), sp(N), E_6, E_7, E_8, F_4 and G_2 cases, the 'double interpretation' of the YT fails in the following sense: the YT obtained according to the Littlewood-Richardson rule as for su(N) now describe reducible representations. For example, if □ denotes the fundamental IR of so(N) we have

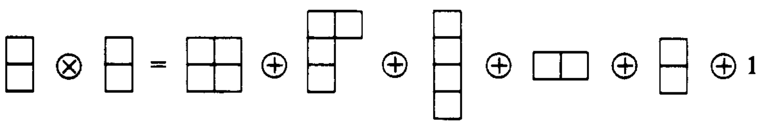


where □□ denotes a direct sum of a $(\frac{1}{2}N(N+1)-1)$-dimensional IR ($\equiv$ □□$_{\mathrm{IR}}$), whose basis vectors are given by $t^i t^j + t^j t^i - (1/N)(\Sigma_{k=1}^{N} t^k t^k)$, and a one-dimensional IR, spanned by the vector $(1/N)(\Sigma_{k=1}^{N} t^k t^k)$ which is obtained from the symmetrised product by contraction of the indices. Therefore the YT of this last representation is obtained from □□ by deleting the two boxes, $1 \equiv$ ⊠⊠.

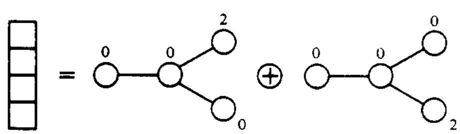
Note that if complete reducibility did not hold and if the 1 were the only ideal in the symmetrised product □ $\overset{S}{\otimes}$ □, □□ should have been denoted by □□$_{\mathrm{IR}}$ ⊕ 1, instead of □□$_{\mathrm{IR}}$ ⊕ 1, and written as ⧄⧄ in the spirit of the present paper (§ 6).

From these remarks, it is apparent that □□, without a subscript, may have two different meanings: □□$_{\mathrm{IR}}$ (referred to as the tensor interpretation (T-YT)), or □□$_{\mathrm{R}}$ (highest-weight interpretation (HW-YT)). In general, only the tensor interpretation of the YT is used. In fact, no confusion may arise in the LA case, due to the complete reducibility. For the same reason, the interpretation in terms of highest weights of these YT are completely understood and unambiguous, allowing another kind of 'double interpretation'.

Although the reductions of the tensor products are more difficult than for su(N), there are now very well established rules available for so(N) and sp(N) (Black *et al* 1983, Dehuai *et al* 1981) (and to a more limited extent for exceptional LA (Bowick and Wybourne 1977, King and Al-Qubanchi 1981)) and can be used for quite substantial representations by hand. For example, concerning so(N), $\forall N$, we have

 (1.1)

where in principle the T-YT describe IR, with some exceptions like



in the so(8) case.

Remark. The decreasing number of boxes on the RHS of (1.1) can be seen as generalised contractions of pairs of tensorial indices. From that point of view, the interpretation of the tensor products for spinor representations requires the use of more sophisticated YT (Fischler 1981, Girardi *et al* 1982, 1983, King and El-Sharkaway 1983).

Since LSA are more complicated than LA, the possibility of defining YST fully compatible with the two above-mentioned interpretations seems improbable. Due to the difficulties encountered in a complete description of the supertensor products, as is seen for the type-1 LSA in § 8; YST are realised here in the 'notation interpretation' spirit (HW-YST) until that section.

A second reason for this choice was to know if it is possible, through a graphical notation, to obtain a particular feeling about the characterisation of the important class of atypical representations which are known to have very specific highest weights. To use YST for such a purpose is quite natural, provided it is possible to define YST satisfying the following demands.

(i) As for the YT, the highest weight should be easily recognised.

(ii) The description of the representation has to be completely determined by the YST, which implies an easy determination of the possible invariant subspaces.

(iii) These YST should help the comprehension of a special property for the so-called type-2 LSA, namely the consistency conditions which necessarily should be satisfied by the HW.

(iv) To be as suggestive as possible of the characteristics of the supertensorial indices which describe the representations.

YST satisfying all these conditions for su(m/n) and osp(m/n), and many of the conditions for the exceptional LSA, are defined throughout §§ 3, 6 and 9, thus giving a positive answer to the above-mentioned question.

In § 2 we recall what the basic LSA are and how their highest weight-maximal representations (HW-MR) are made. In § 3 we propose Young supertableaux describing these representations in the su(m/n) and osp(m/n) cases. In § 4 we give some ways to determine the possible ideals of such representations. In § 5 we describe SRR which can be defined only through a generalised notion of highest weight. We call them the pseudo-highest-weight representations (PHW representations). In § 6 we have successfully established in a systematic way the correspondence between all these representations and supertableaux. In § 7 we present briefly some results which may be obtained with the help of the (P)HW-YST. In § 8 a new kind of YST is introduced which is apparently better adapted for the illustration of some tensor product rules than are the (P)HW-YST. In § 9 we define YST for the exceptional LSA.

We also present in the conclusions some considerations on strange and Cartan-type LSA, and we speculate about the existence of YST for a class of more exotic SRR than the ones defined in § 5.

Finally, the tables are collected in appendix 1.

2. Maximal representations with highest weight

2.1. *Simple Lie superalgebras*

A LSA is by definition a Z_2-graded LA, $G \equiv G_{\bar{0}} + G_{\bar{1}}$, for which the following commutation relations hold:

(i) $[G_{\bar{0}}, G_{\bar{0}}] \subset G_{\bar{0}}$; thus $G_{\bar{0}}$ is an ordinary LA;

(ii) $[G_{\bar{0}}, G_{\bar{1}}] \subset G_{\bar{1}}$; thus G_1 is a $G_{\bar{0}}$ module;

(iii) $\{G_{\bar{1}}, G_{\bar{1}}\} \subset G_{\bar{0}}$, where { , } means the anticommutator.

As for the Lie algebras, G is said to be simple if it contains no non-trivial ideal. All the complex simple LSA have been classified (Kac 1977, Scheunert 1979), and as such are listed in table A1.

The few isomorphisms among simple LSA are given in table A2, and some other notations that we encounter in the literature are shown in table A3.

In this paper we are only interested in the basic LSA, table A4. Essentially, the basic LSA are analogous to the finite simple LA. In these cases the Cartan subalgebra H of G coincides with that of $G_{\bar{0}}$ and we have $G = H \oplus (\oplus\ G_\alpha)$ (where $\dim G_\alpha = 1\ \forall \alpha$, except for $A(1, 1)$) and $\Delta \equiv \Delta_0 + \Delta_1$ is the root system of G.

The Cartan-type LSA are the finite analogues of infinite-dimensional Cartan–Lie algebras, and the simple LSA which have no direct LA analogue are called 'strange' (although the $Q(n)$ may be the finite analogues of the loop algebras $\mathrm{su}(n) \otimes C[x, x^{-1}]$).

Before revealing some additional information about the structure of the basic LSA in table A4, clarification of certain points is needed.

(i) The superdimension of a Z_2-graded space is, by definition, the dimension of the even space minus the dimension of the odd one; for example, $\mathrm{sdim}\, G = \dim G_{\bar{0}} - \dim G_{\bar{1}}$; and when V denotes a representation of G, $\mathrm{sdim}\, V = \dim V_{\bar{0}} - \dim V_{\bar{1}}$.

(ii) When an odd space $G_{\bar{1}}$ is completely reducible (irreducible) the LSA is said to be of type 1 (type 2).

A summary of the properties of the roots, simple roots, Dynkin diagrams and Cartan matrices of these LSA is given in tables A6 and A7. Precise definitions of these objects, bilinear forms and other definitions were given previously in Kac (1977, 1978). We simply recall the following.

(*a*) A particular property of the standard weights ε_i, δ_j is $\mathrm{sgn}(\varepsilon_i, \varepsilon_i) = -\mathrm{sgn}(\delta_j, \delta_j)$. This implies that the norm of some odd roots may be degenerate: $\bar{\Delta}_1 \equiv \{\alpha \in \Delta_1 | (\alpha, \alpha) = 0\}$. In the case of $\mathrm{osp}(1/2n)$ and $G(3)$, there exists odd roots α such that $2\alpha \in \Delta_0$, thus implying $(\alpha, \alpha) \neq 0$ for them.

(*b*) In a Kac–Dynkin diagram the simple even roots $\alpha \in \Delta_0$ are denoted by a traditional white circle ○, by a grey one ⊗ if $\alpha \in \bar{\Delta}_1$, or by a black one ● if $\alpha \in (\Delta_1 \backslash \bar{\Delta}_1)$.

(*c*) For a given basic LSA there may exist many systems of simple roots inequivalent under Weyl reflections, but it is always possible to find a system of simple roots with only one odd root called β in table A6 and α_s in the rest of this paper. In this paper the Weyl group is taken to be that of $G_{\bar{0}}$, although a consistent definition of a Weyl group for the whole LSA was given recently (Leites *et al* 1985). In particular $D(m, n)$ and $C(m, n)$ will sometimes denote in this paper the two Weyl-inequivalent systems of simple roots of $\mathrm{osp}(2m/2n)$.

(*d*) The standard definitions of the Cartan matrices and Dynkin diagrams of $\mathrm{su}(n)$, $\mathrm{so}(2n+1)$, $\mathrm{sp}(2n)$, $\mathrm{so}(2n)$ and G_2 are stated in table A5. They are part of the corresponding definitions for the LSA in tables A6 and A7.

The relationship between any of the above-mentioned Cartan matrices A with entries $a_{ij} = (A)_{ij}$ and the corresponding simple root system is given by $a_{ij} \equiv 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. For the LSA other than $\mathrm{osp}(1/N)$ this definition has to be modified when the simple odd root is involved: $a_{is} \equiv 2(\alpha_i, \alpha_s)/(\alpha_i, \alpha_i)$ again if $i \neq s$, but in any case $a_{si} \equiv (\alpha_s, \alpha_i)$. Since $a_{ss} = 0$, we are free to choose the normalisation $a_{s,s+1} = +1$.

Remark 1. $\mathrm{su}(n/n)$ is the central extension of $A(n-1, n-1) = \mathrm{su}(n/n)/Z_{2n}$ where Z_{2n} is the ideal generated by 1_{2n}. The next considerations about $\mathrm{su}(m/n)$ IR and SRR, $m \neq n$, do apply also for $\mathrm{su}(n/n)$, but do not apply for $A(n-1, n-1)$, whose representations will not be studied in this paper.

2.2. *Representations of the basic Lie superalgebras*

Let H_{α_i}, $E_{+\alpha}$, $E_{-\alpha}$ be the generators of a LSA where the H_{α_i} form a basis of the Cartan subalgebra H. The $E_{+\alpha}(E_{-\alpha})$ act as raising (respectively lowering) operators and the commutation relations among these generators are given by

$$[H_{\alpha_i}, H_{\alpha_j}] = 0 \qquad [E_{\alpha_i}, E_{-\alpha_j}\} = \delta_{ij} H_{\alpha_j} \qquad \forall \alpha_i, \alpha_j$$

$$[H_{\alpha_i}, E_{\pm\alpha_j}] = \pm a_{ij} E_{\pm\alpha_j} \qquad \text{where the } a_{ij} \text{ are the components of the Cartan matrix} \tag{2.1}$$

$$[E_\alpha, E_{\alpha'}\} = \begin{cases} N(\alpha, \alpha') E_{\alpha+\alpha'} & \text{if } \alpha+\alpha' \in \Delta \\ 0 & \text{otherwise.} \end{cases}$$

Then a representation $V(\Lambda)$ with highest weight $\Lambda \in H^*$ are characterised in the following way. There is a particular vector $|\Lambda\rangle \in V(\Lambda)$ such that

$$E_{+\alpha}|\Lambda\rangle \equiv 0 \qquad \forall \alpha \in \Delta_+, \text{ the set of positive roots} \tag{R1}$$

$$H_{\alpha_i}|\Lambda\rangle \equiv a_i|\Lambda\rangle \tag{R2}$$

and we get all the remaining vectors spanning the representation by applying successively the lowering generators $E_{-\alpha}$ to $|\Lambda\rangle$.

The Dynkin indices a_i corresponding to the simple roots α_i are defined as

$$a_i = 2(\Lambda, \alpha_i)/(\alpha_i, \alpha_i) \tag{2.2}$$

$$a_s = (\Lambda, \alpha_s) \tag{2.3}$$

where α_s is the only simple odd root.

The values of the $a_{i\neq s}$ are directly related to the dimensionality of $V(\Lambda)$:

$$a_i \notin Z_+ \cup \{0\} \qquad \text{implies} \qquad (E_{-\alpha_i})^k|\Lambda\rangle \neq 0 \ \forall k \geqslant 0 \tag{2.4}$$

and hence $V(\Lambda)$ is infinite dimensional.

$$(E_{-\alpha_i})^k|\Lambda\rangle \neq 0 \qquad \text{and} \qquad (E_{-\alpha_i})^{k+1}|\Lambda\rangle = 0 \qquad \text{implies} \qquad k = a_i \tag{2.5}$$

and thus $a_i \in Z_+ \cup \{0\}$ is clearly required for finite dimensionality.

More generally the commutation relations (2.1) and (2.5) imply (2.6):

$$(E_{-\alpha})^{k(\alpha)}|\Lambda\rangle = 0 \tag{2.6}$$

where $k(\alpha) = 2(\Lambda+\rho, \alpha)/(\alpha, \alpha)$ and where $\rho = \rho_0 - \rho_1$ is defined in such a way that

$$(\rho, \alpha_i) \equiv \tfrac{1}{2}(\alpha_i, \alpha_i) \tag{2.7}$$

in particular $(\rho, \alpha_s) = 0$ (explicitly $\rho_p = \frac{1}{2}\Sigma_{\alpha\in\Delta_p} \alpha$, $p = 0, 1$).

The value of a_s is related to the atypicality of $V(\Lambda)$: using (2.2) (2.3) and (2.7) we can compute the set N of non-typical values (table A11):

$$N \equiv \{a_s = a_s(a_{i\neq s}) \text{ such that } (\Lambda+\rho, \alpha) = 0 \text{ for some } \alpha \in \Delta_1\}.$$

For the type-1 LSA, since $\bar{\Delta}_1$ coincide with Δ_1, we have $\{E_{-\alpha}, E_{-\alpha}\} = 0$, hence $(E_{-\alpha})^2|\Delta\rangle = 0$, $\forall \alpha \in \Delta_1$. Therefore the analogues of (2.4)-(2.6) are

if $a_s \notin N$, i.e. $V(\Lambda)$ is typical, then $E_{-\alpha}|\Lambda\rangle \neq 0$, $\forall \alpha \in \Delta_1$ (however, $E_{-\alpha}|\Lambda\rangle \neq 0$, $\forall \alpha \in \Delta_1$ does not imply always typicality) (2.4′)

$E_{-\alpha_s}|\Lambda\rangle = 0$ implies $a_s = 0$, this condition coming from $(\Lambda+\rho, \alpha_s) = 0$. (2.5′)

From (2.4′) we immediately obtain

$$\text{if there is an } \alpha \in \Delta_1 \text{ such that } E_{-\alpha}|\Lambda\rangle = 0, \quad \text{then } a_s \text{ allow } (\Lambda+\rho, \alpha') = 0 \text{ for some } \alpha' \in \Delta_1. \tag{2.6′}$$

For the type-2 LSA, the situation is not so simple; however, the important point is that any $E_{-\alpha}|\Lambda\rangle = 0$ implies a value for a_s which is called atypical, as for type-1 LSA. In both cases the converse is not true: given a non-typical a_s, there can exist a non-zero vector $|\chi\rangle \in V(\Lambda)$ that we obtain by applying the odd generators $E_{-\alpha}$ to $|\Lambda\rangle$ such that $E_{+\alpha_i}|\chi\rangle = 0$, $\forall \alpha_i$. Hence χ is the highest weight of an invariant subspace $I(\Lambda) \subset V(\Lambda)$.

In fact a quite similar phenomenon already occurs in the LA case: let the non-negative integers a_i determine, say, $V_0(\Lambda)$. If some $|\chi\rangle \equiv (E_{-\alpha_i})^{a_i+1}|\Lambda\rangle$ does not vanish, then $E_{+\alpha}|\chi\rangle \equiv 0$ again, $\forall \alpha$. This implies the existence of a maximal invariant subspace $I_0(\Lambda)$ of $V_0(\Lambda)$. In contrast to the LSA case, both $V_0(\Lambda)$ and $I_0(\Lambda)$ are necessarily infinite, however, in such a way that $V_0(\Lambda)_{\mathrm{IR}} = V_0(\Lambda)/I_0(\Lambda)$ is the unique finite representation specified by the a_i.

As an alternative to the rules (R1) and (R2), the above superrepresentation $V(\Lambda)$ can be characterised by the same rule (R1) but where the α_i are now the simple roots of $G_{\bar{0}}$; while in place of (R2), $H_{\alpha_s}|\Lambda\rangle = a_s|\Lambda\rangle$ we have to consider the rules (R2′):

$Q|\Lambda\rangle = q|\Lambda\rangle$, Q being, for the type-1 LSA other than su(n/n), the u(1) generator commuting with the whole even subalgebra, or

$C|\Lambda\rangle = c|\Lambda\rangle$, C being the centre of su(n/n), or (R2′)

$H_\delta|\Lambda\rangle = b|\Lambda\rangle$, H_δ being, for the type-2 LSA, the generator of the Cartan subalgebra corresponding to the $G_{\bar{0}}$-simple root δ which is not a G-simple root. Thus b also has to be a non-negative integer for having finite $V(\Lambda)$.

The expressions for these generators are given in table A8.

In particular, we see that for the type-2 LSA (some $D(2, 1, \alpha)$ excepted), a_s is necessarily a non-negative rational number. For these LSA, some additional constraints on the Dynkin indices (table A9) appear occasionally. The origin of these constraints, also called 'consistency conditions', will be explained later.

In contrast any complex number is allowed for q or c, each value being known to specify a particular one-dimensional IR of the Abelian part of the even subalgebra. Thus there is no need of additional restrictions on the a_s for the type-1 LSA, the representations defined in this manner also being finite because of the nilpotency of G_{-1}.

The atypicality conditions arising from $(\Lambda+\rho, \alpha) = 0$ are sometimes easier to get when the odd roots α, given previously in tables A6 and A7 in terms of the fundamental weights ε_i, δ_j, are given (table A10) in terms of the simple roots α_i. In particular, table A11 gives the general expressions for these non-typicality conditions in terms of the Dynkin indices.

For the type-1 LSA, each root α provides a non-typicality condition in every case. For the type-2 LSA, it is even more comprehensive to express the non-typicality conditions in terms of the Dynkin indices of the even subalgebra, as is done in table A12.

(i) Consequently there can be no non-typicality conditions associated with the roots $\alpha = \delta_i$ of $B(m, n)$, or $\alpha = \delta$ of $G(3)$, a result obvious from (2.6) since $2(\Lambda, 2\alpha)/(2\alpha, 2\alpha) = k(2\alpha) \in Z_+ \bigcup \{0\}$. Thus $2(\Lambda+\rho, \alpha)/(\alpha, \alpha) \geq \frac{1}{2}k(2\alpha) + 1 > 0$, whatever the above α stand for.

(ii) This is again the case with the roots $\alpha = \delta_i - \varepsilon_j$ of $B(m, n)$, $D(m, n)$, $D(2, 1, \alpha)$ and $G(3)$, when the values of b do not imply the consistency conditions. For $F(4)$, the same effect happens when considering the roots $\delta \pm \varepsilon_1 + \varepsilon_2 + \varepsilon_3$ and $\delta + \varepsilon_1 - \varepsilon_2 + \varepsilon_3$. However, for small values of b, these roots can provide accidental non-typicality conditions.

(iii) In constrast, no general considerations forbid the non-typicality conditions associated with the roots of the $\delta_i + \varepsilon_j$ type ($\delta \pm \varepsilon_1 \pm \varepsilon_2 - \varepsilon_3$ and $\delta - \varepsilon_1 - \varepsilon_2 + \varepsilon_3$ for $F(4)$). Therefore, only these non-typicality conditions are the true analogues of the type-1 LSA non-typicality conditions.

2.3. Maximal representations of the type-1 LSA

In this subsection we indicate briefly how to build the unique maximal representation (MR) specified by a HW in the type-1 LSA case.

The Z_2-graded type-1 LSA $G \equiv G_{\bar{0}} + G_{\bar{1}}$ have furthermore an additional consistent Z grading, in particular the following one:

$$G = \bigoplus_{i=-1}^{1} G_i \qquad \text{with} \qquad [G_i, G_j\} \subset G_{i+j} \tag{2.8}$$

where $G_0 \equiv G_{\bar{0}}$ and the roots of G_{+1} (respectively G_{-1}) are the positive (respectively negative) odd roots.

By our definition of Λ we have $E_\alpha|\Lambda\rangle \equiv 0 \; \forall \alpha \in \Delta_+$. Thus in particular we have $G_{+1}|\Lambda\rangle = 0$. The even generators applied to $|\Lambda\rangle$ give, in the standard way, the G_0 representation that we call $V_0(\Lambda)$. We note that $G_{+1} V_0(\Lambda) = 0$ as $[G_0, G_{+1}] \subset G_{+1}$. Then $V_0(\Lambda)$ can also be seen as a kind of non-faithful IR of the non-semisimple LSA $P \equiv G_0 \oplus G_{+1}$. Therefore, on $V_0(\Lambda)$, as we have $\{G_{-1}, G_{-1}\} \equiv 0$, we can only apply a completely antisymmetric combination of the odd generators belonging to G_{-1}, giving maximally the following representation of G:

$$V(\Lambda)_{\text{MR}} = V_0(\Lambda) \otimes \left(\sum_{k=0}^{N} \Lambda^k G_{-1} \right) \qquad N = \dim G_{-1} \tag{2.9}$$

built from $V_0(\Lambda)$ by induction of the generators of G/P; hence the notation $V(\Lambda) = \text{Ind}_P^G \, V_0(\Lambda)$ in Kac (1978).

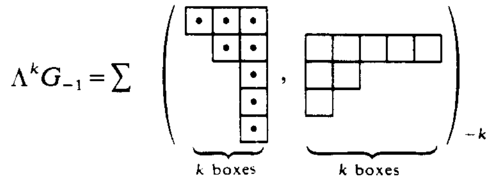
$\Lambda^k G_{-1}$ means the kth completely antisymmetric tensor product of G_{-1} with itself. Thus $\dim G_{-1} = N$ implies $\dim(\Lambda^k G_{-1}) = \binom{N}{k}$. Therefore we obviously have for type-1 LSA

$$\dim V(\Lambda)_{\text{MR}} = \sum_{k=0}^{N} \binom{N}{k} \dim V_0(\Lambda) = 2^N \dim V_0(\Lambda) \qquad N = \begin{cases} mn & \text{for } \mathrm{su}(m/n) \\ 2n & \text{for } \mathrm{osp}(2/2n) \end{cases}$$

$$\text{sdim } V(\Lambda)_{\text{MR}} = \sum_{k=0}^{N} (-1)^k \binom{N}{k} \dim V_0(\Lambda) = 0. \tag{2.10}$$

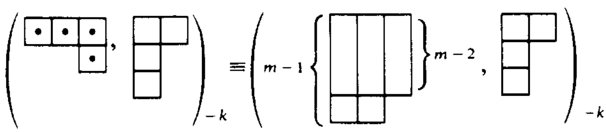
Definition. The YT corresponding to an irreducible representation of the LA $\mathrm{su}(m) + \mathrm{su}(n) + \mathrm{u}(1)$ is given by two YT, each one corresponding to a $\mathrm{su}(N)$ factor, and an eigenvalue for the $\mathrm{u}(1)$. Similarly the YT of a $\mathrm{sp}(2n) + \mathrm{u}(1)$-IR is the sum of a $\mathrm{sp}(2n)$-YT and a number.

In terms of the G_0-YT we have, more explicitly, for su(m/n):



(2.11)

where the sum is taken over all the inequivalent pairs of tableaux, each having k boxes and the su(m) tableau being the su(n) one, transposed. The meaning of the dot in the boxes ⊡ is explained in § 1. Thus the columns of l dotted boxes are equivalent to the columns of $(m-l)$ undotted boxes, e.g.

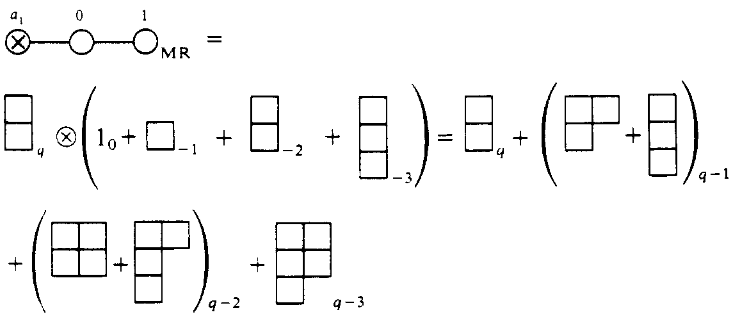


(2.12)

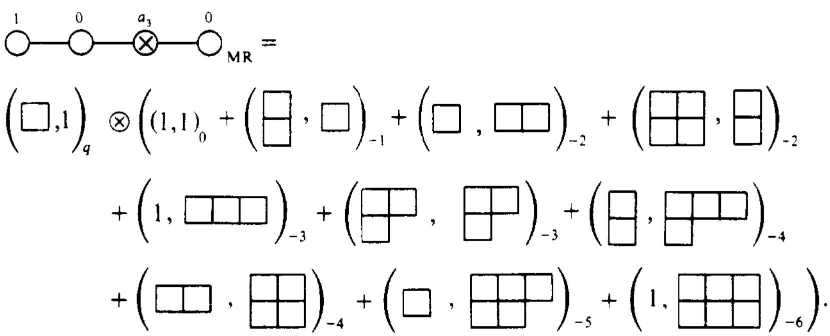
$(-k)$ is the integer we have to add to the q value of the highest weight since we have

$$[Q, E_{\pm\alpha}] = \begin{cases} 0 & \text{when } \alpha \in \Delta_0 \\ \pm E_{\pm\alpha} & \text{when } \alpha \in \Delta_1 \end{cases}.$$

Therefore the Z grading of the LSA induces a Z grading of the representation. Some examples are

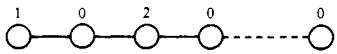


(2.13)

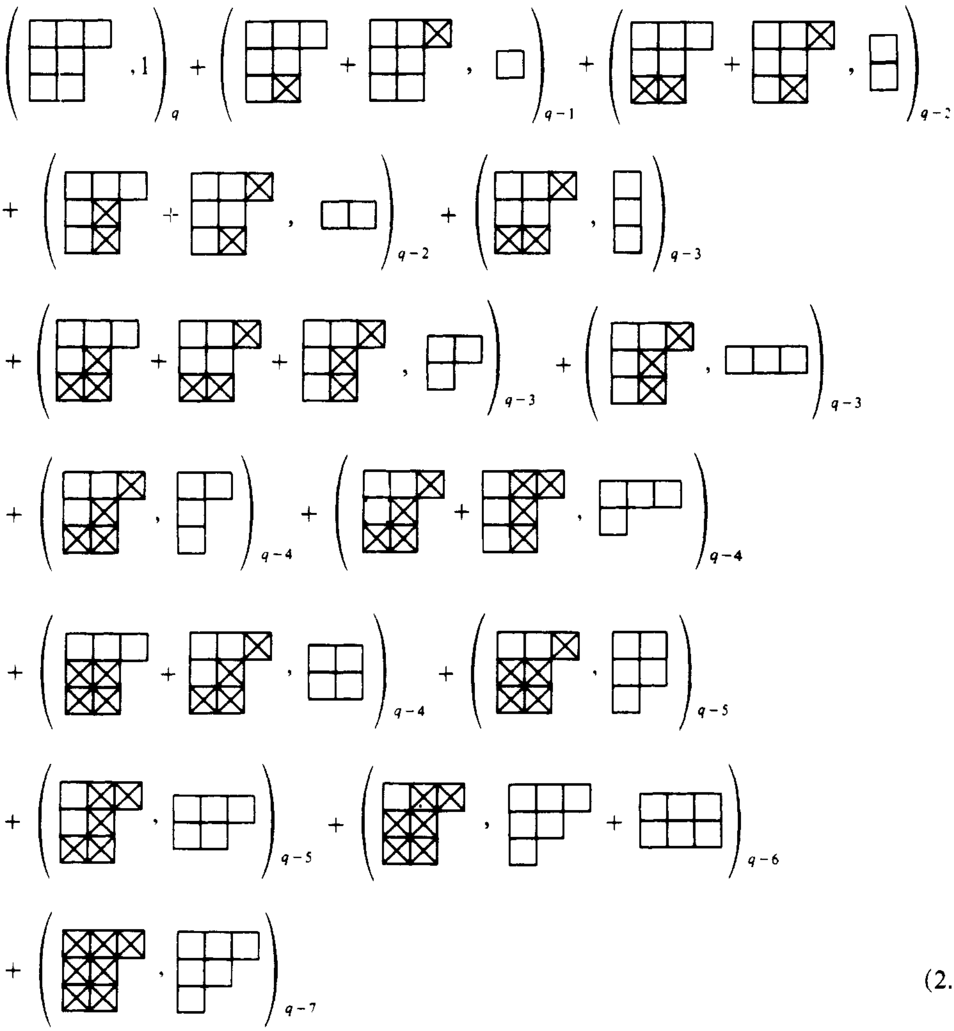


(2.14)

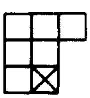
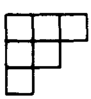
Alternatively, we can see su(m/n) as a supersymmetric analogue of su($m+n$) and supersymmetrise the usual branching of su($m+n$) into su(m)+su(n)+u(1) (Bars *et al* 1983). However, we do not usually obtain $V(\Lambda)_{\mathrm{MR}}$ with this procedure; for example, the su($3/M$) supersymmetrised version of the su($3+M$) representation



leads to the following $V(\Lambda)_{\mathrm{IR}}$:

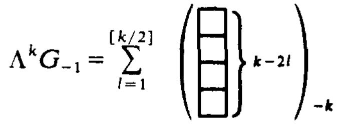


(2.15)

where  means here , and so on.

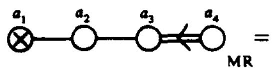
For su(n/n) the only difference from su(m/n) is that all the su(n)+su(n)+u(1) multiplet have the same c value, C, commuting with the whole superalgebra.

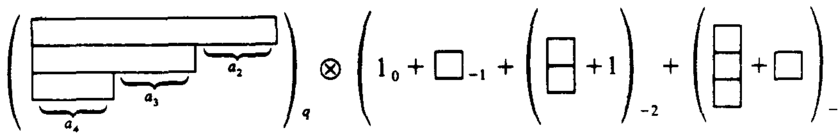
For osp(2/2*n*)

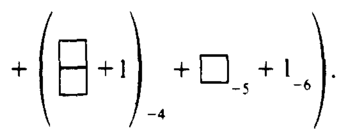


(2.16)

where $[k/2]$ denotes the integer part of $k/2$. Therefore



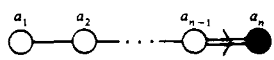
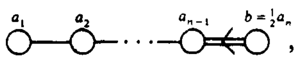


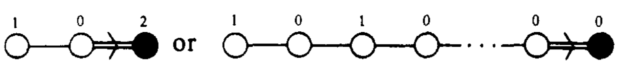


(2.17)

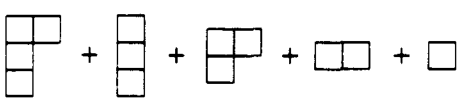
2.4. Representations of osp(1/2n)

The osp(1/2*n*) is the only class among all the simple LSA that does not possess non-typical representations. For that reason complete reducibility holds for their representations.

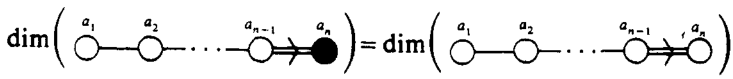
The $G_{\bar{0}} = \mathrm{sp}(2n)$ content of $V(\Lambda) =$  is easy to exhibit: given the YT of $V_0(\Lambda) =$ , the action of G_{-1} on V_0 is to destroy a box of the starting YT. Since $\{G_{-1}, G_{-1}\} \subset G_{\bar{0}}$ we must keep all the sp(2*n*)-YT obtained from that of $V_0(\Lambda)$ by removing at most one box from each row, e.g.



gives

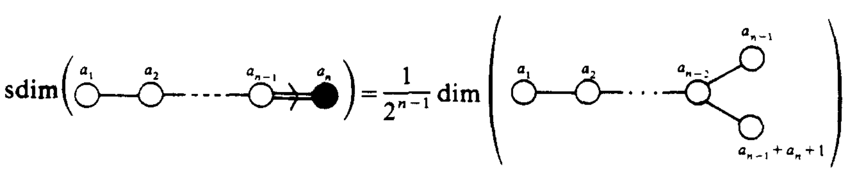


of sp(2*n*). It is easy to verify that



(2.18)

the one-to-one correspondence between weights of osp(1/2*n*) and so(2*n*+1) IR being well known (Rittenberg and Scheunert 1982). We can calçulate from Kac (1978) that



(2.19)

2.5. *Maximal representations of Osp(M/2n), M ≥ 3*

Considering the former osp$(1/2n)$ example, we remark that $V(\Lambda)$ is no longer constructed by the usual induction of G_{-1} on $V_0(\Lambda)$. In fact, we have implicitly followed the theorem for the realisation of the type-2 MR (Kac 1977):

$$V(\Lambda)_{\mathrm{MR}} = \mathrm{Ind}_P^G\, V_0(\Lambda)/U(G)G_{-\delta}^{k+1}V_0(\Lambda). \tag{2.20}$$

The ideal $M(\Lambda) \equiv U(G)G_{-\delta}^{k+1}V_0(\Lambda)$ is non-trivial only when the Dynkin index b takes some specific values (see table A9). To be precise, when $b \leq [M/2]$, both $\mathrm{Ind}_P^G\, V_0(\Lambda)$ and $U(G)G_{-\delta}^{k+1}V_0(\Lambda)$ are infinite, as we shall see later.

The first difficulty is to express $\mathrm{Ind}_P^G\, V_0(\Lambda) \equiv (*)$ in terms of $G_{\bar{0}}$ representations as the G_{-1}, present in the formula $V_0(\Lambda)\otimes(\Sigma_{k=0}^{M}\Lambda^k G_{-1})$, does not by itself give a $G_{\bar{0}}$ representation; it is the whole $G_{\bar{1}}$ $(\equiv G_{+1}\oplus G_{-1})$ which gives the fundamental $G_{\bar{0}}$ irreducible representation. In fact, $(*)$ is a subspace of

$$(**) \equiv V_0(\Lambda)\otimes\left(\sum_{k=0}^{M}\Lambda^k G_{\bar{1}}\right). \tag{2.21}$$

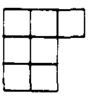
Definition. The operator $T = \Sigma_{k=1}^{n-1} k\cdot h_k + n\cdot H_\delta$ when applied to $|\Lambda\rangle$ counts the number of boxes in the sp$(2n)$ part of the YT specifying $V_0(\Lambda)$.

Lemma. $|\Lambda\rangle$ belongs to the subspace specified by the so(M)+sp$(2n)$-YT which has the most boxes in its sp$(2n)$ part.

Proof. We have $[T, E_{\pm\alpha}] = \pm E_{\pm\alpha}$ $\forall\alpha\in\Delta_1$. Thus T plays a role similar to Q for the type-1 LSA (except now that $[T, E_{\pm 2\alpha}] = \pm 2E_{\pm 2\alpha}$ when $2\alpha\in\Delta_0$). As for Q we have $[T, X] = 0$, $\forall X = E_{\mu\neq 2\alpha}\in G_{\bar{0}}$. Let $T|\Lambda\rangle = t|\Lambda\rangle$, then the other eigenvalues of T can only be smaller, the corresponding eigenvectors being obtained by applications of the $E_{-\alpha}$ on $|\Lambda\rangle$.

This operator will help us to select inside $(**)$ a subspace containing $(*)$ as we shall see in the following example where we are interested in the osp$(M/2n)$-MR, $V(\Lambda)_{\mathrm{MR}}$, specified by the following sp$(2n)$+so(M)-IR:

$V_0(\Lambda) =$ (

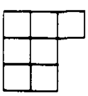


, 1). (2.22)

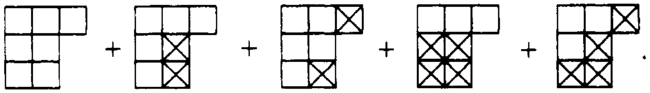
There is a natural embedding of the $(M+2n)$-dimensional osp$(M/2n)$-IR into the standard su$(M/2n)$-IR one. Therefore the su$(M/2n)$-IR (2.15), which we shall call W, is also an osp$(M/2n)$ representation but where the su(M) and su$(2n)$-YT are now describing reducible so(M) and sp$(2n)$ representations.

In particular a su$(2n)$-IR, whose YT is made of k boxes, decomposes into many sp$(2n)$-IR, whose respective YT are made of $k-2j$ boxes, $j = 0, 1, \ldots, [k/2]$ (a similar phenomenon also appears when the branching of su(M) into so(M) is considered). Hence, T separates W into orthogonal subspaces according to the branching of the

su(2n)-IR



into sp(2n)-IR:

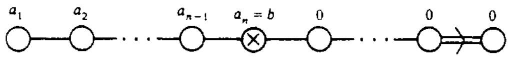


Let $\tilde{V}(\Lambda)$ denote the set of YT which looks like (2.15), but where now the left boxes describe sp$(2n)$-IR and the right boxes continue to describe the previous reducible so(M) representations (the u(1) factor is irrelevant here). Thus $\tilde{V}(\Lambda)$ is therefore clearly a subspace of W, and furthermore is also a subspace of (**).

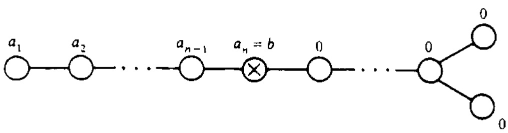
In particular we note that inside W the above-mentioned so(M)+sp$(2n)$ representation $V_0(\Lambda)$ is the one having the highest T eigenvalue. Therefore W necessarily contains at least $V(\Lambda)_{\mathrm{IR}}$ according to the lemma.

Lemma 2. In the typical case, $\tilde{V}(\Lambda)$ coincides with $V(\Lambda)_T$.

Proof. The above arguments may be generalised for any $V_0(\Lambda)$ where the so(M) part is trivial, giving a corresponding $\tilde{V}(\Lambda)$ in the general case of

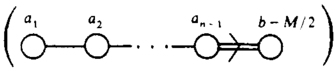


or



we can then check that dim $\tilde{V}(\Lambda) \geq \mathcal{D}$, where

$$\mathcal{D} \equiv 2^{nM} \dim\left(\quad \right) \qquad (2.23)$$



The equality holds if and only if $b \geq M-1$; in that case, we also have sdim $\tilde{V}(\Lambda) = 0$.

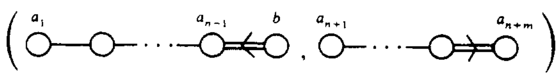
But if $b \geq M$, the representation is typical and we know by Kac's formula that dim $V(\Lambda) = \mathcal{D}$ and sdim $V(\Lambda) = 0$. Therefore we conclude that in the typical case $\tilde{V}(\Lambda) = V(\Lambda)$.

If $b < M$ and all the so(M) indices a_i are zero, then the representation is non-typical. The complete reducibility of $\tilde{V}(\Lambda)$ seems to hold in this case: $\tilde{V}(\Lambda) = V(\Lambda)_{\mathrm{IR}} \oplus V$ implying $V(\Lambda)_{\mathrm{MR}} \approx V(\Lambda)_{\mathrm{IR}}$.

Remark. Contrary to the type-1 LSA, where in the atypical case dim $V(\Lambda)_{\mathrm{IR}} <$ the typical dimension formula, for type-2 LSA we can have dim $V(\Lambda)_{\mathrm{IR}} > \mathcal{D}$. This is fortunate since we would get $\mathcal{D} < 0$ when $b < M/2$.

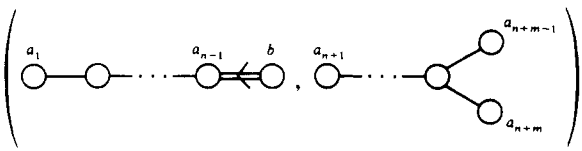
Generally, when the so(M) part is non-trivial:

$$V_0(\Lambda) \equiv (V_{\mathrm{sp}(2n)}, V_{\mathrm{so}(2m+1)}) \equiv$$



or

$$V_0(\Lambda) \equiv (V_{\mathrm{sp}(2n)}, V_{\mathrm{so}(2m)}) \equiv$$



the dimension formulae for typical representations

$$\dim V(\Lambda) \equiv \mathscr{D} \cdot \dim V_{\mathrm{so}(M)} \tag{2.24}$$

$$\operatorname{sdim} V(\Lambda) \equiv 0 \tag{2.25}$$

imply

$$V(\Lambda)_{\mathrm{MR}} \equiv \tilde{V}(\Lambda) \otimes (1, V_{\mathrm{so}(M)}) \qquad \text{if } b \geqslant M-1.$$

If $M/2 < b < M-1$, then $V(\Lambda)$ can be typical. Thus we have $\dim V(\Lambda)_T < \mathscr{D} \cdot \dim V_{\mathrm{so}(M)}$. This implies that

$$\bar{V}(\Lambda) \equiv \tilde{V}(\Lambda) \otimes (1, V_{\mathrm{so}(M)})$$

is completely reducible: $\bar{V}(\Lambda) = V(\Lambda)_T \oplus V'$. Similarly, in the non-typical case we have $\bar{V}(\Lambda) = V(\Lambda)_{\mathrm{MR}} \oplus V'$ when $b < M-1$.

Note that V' in general is not an osp(M/N) representation, but a finite-dimensional subspace of $\bar{V}(\Lambda)$, which has to be merged with some infinite-dimensional so(M)+sp(N) subspaces of $\operatorname{Ind}_P^G V_0(\Lambda)$ in order to make up $M = U(G) G_{-\delta}^{k+1} V_0(\Lambda)$. The origin of these infinite subspaces in $\operatorname{Ind}_P^G V_0(\Lambda)$ is the following.

Firstly, let $\dot{\Delta}_1$ denote the set of roots of the $\delta_n - \varepsilon_i$ type, then we have

$$[H_\delta, E_\alpha] = \begin{cases} \pm E_{\pm\alpha} & \forall \alpha \in \dot{\Delta}_1 \\ 0 & \text{if } \alpha \in \Delta_1 - \dot{\Delta}_1. \end{cases} \tag{2.26}$$

Therefore each $E_{-\sigma_i} \equiv E_{-(\delta_n - \varepsilon_i)}$ subtracts one unit from b when applied to any $|\Gamma\rangle$.

Secondly, the vector $|\chi\rangle \equiv E_{-\sigma_b} E_{-\sigma_{b-1}} \ldots E_{-\sigma_2} E_{-\sigma_1} |\Lambda\rangle$ which clearly belongs to $\operatorname{Ind}_P^G V_0(\Lambda) \equiv V_0(\Lambda) \otimes (\Sigma_{k=0}^{M} \Lambda^k G_{-1})$ satisfies $E_{+\alpha}|\Lambda\rangle = 0$, $\forall \alpha \in \Delta_0$. Thus χ is also a sp$(2n)$-HW. However, according to (2.26), its Dynkin index $b = 2(\chi, \delta)/(\delta, \delta) = -1$. Then $\dim \operatorname{Ind}_P^G V_0(\Lambda) = \infty$. Thus it is necessary that $E_{+\alpha_i}|\chi\rangle = 0$ in order for χ to be the highest weight of the infinite-dimensional invariant subspace $M = U(G) G_{-\delta}^{b+1} V_0(\Lambda)$. Indeed, if $E_{+\alpha}|\chi\rangle \neq 0$, then $|\chi\rangle$ belongs to $V(\Lambda)_{\mathrm{IR}}$, implying unavoidably $\dim V(\Lambda)_{\mathrm{IR}} = \infty$.

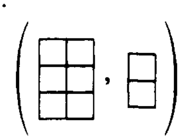
A simple calculation show that $E_{+\sigma_1} E_{+\sigma_2} \ldots E_{+\sigma_b} |\chi\rangle$ is proportional to $a_n(a_n - a_{n+1} - 1) \ldots (a_n - a_{n+1} - \ldots - a_{n+b} - b)|\Lambda\rangle$, i.e. to

$$\left(b + \sum_{i=1}^{m} a_{n+i}\right)\left(b - 1 + \sum_{i=2}^{m} a_{n+i}\right) \ldots \left(1 + \sum_{i=b}^{m} \tilde{a}_{n+i}\right)\left(\sum_{i=b+1}^{m} \tilde{a}_{n+i}\right)|\Lambda\rangle$$

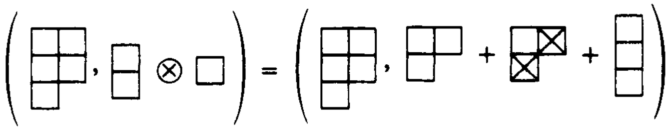
where $\tilde{a}_{n+m} = \frac{1}{2} a_{n+m}$ for $B(m, n)$ or $\tilde{a}_{n+m-1} = \frac{1}{2}(a_{n+m-1} - a_{n+m})$ for $D(m, n)$; otherwise $\tilde{a}_i = a_i$.

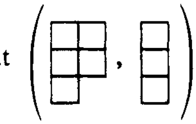
We see that this expression vanishes, allowing finite dimensionality, if and only if the corresponding consistency condition, table A9, holds. In other words, a consistency condition is nothing other than a necessary non-typicality condition in order to have finite-dimensional representations.

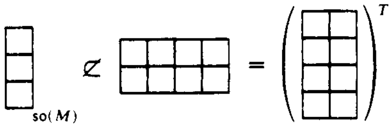
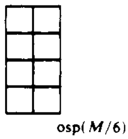
The pieces of $\bar{V}(\Lambda)$ that we should reject in order to get $V(\Lambda)_{\mathrm{MR}}$ are those incompatible with the supersymmetrisation of so$(M+2n)$ (Morel *et al* 1985). In fact, the osp(M/N)-YST are precisely defined in the next section in order to illustrate this incompatibility. The rule is the following: reject a piece if the so(M) part of the YT cannot take place in the osp(M/N)-YST transposed.

The following example is for osp$(M/6)$. If $V_0 \equiv$  of sp(6)+so(M)

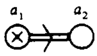
then



is expected to be in $V(\Lambda)_{\mathrm{MR}}$, but  cannot be in $V(\Lambda)_{\mathrm{MR}}$ since

 , where  is the YST corresponding to $V(\Lambda)_{\mathrm{MR}}$.

A more complete treatment of the osp(M/N) representations can be found in Hurni and Morel (1982) and Morel *et al* (1985). We present here only the final results for osp(3/2).

Let  , then $V_0(\Lambda) = (b+1, a_2+1)$ is the $((b+1)\cdot(a_2+1))$-dimensional IR of $\mathrm{su}(2)+\mathrm{su}(2) \approx \mathrm{sp}(2)+\mathrm{so}(3)$, $b = a_1 - \frac{1}{2}a_2$. Then

$$V(\Lambda)_{\mathrm{MR}} = (b+1, a_2+1) + (b, (a_2+3)+(a_2+1)+(a_2-1)) + (b-1, (a_2+3)+(a_2+1)+(a_2-1)) + (b-2, a_2+1). \qquad (2.27)$$

Thus

$$\dim V(\Lambda)_{\mathrm{MR}} = 2^3(b - \tfrac{3}{2}+1)(a_2+1)$$

$$\mathrm{sdim}\ V(\Lambda)_{\mathrm{MR}} = 0.$$

In the non-typical case $a_1 = a_2+1$, i.e. $b = 1 + \frac{1}{2}a_2$, we have in general

$$\begin{aligned} V(\Lambda)_{\mathrm{IR}} &= (b+1, a_2+1) + (b, (a_2+3)+(a_2+1)) + (b-1, a_2+3) \\ I(\Lambda) &= (b, a_2-1) + (b-1, (a_2+1)+(a_2-1)) + (b-2, a_2+1). \end{aligned} \qquad (2.28)$$

Thus

$$\dim V(\Lambda)_{\mathrm{IR}} = 4b(a_2+2) - 2$$

$$\dim I(\Lambda) = 4(b-1)(a_2) - 2$$

$$\mathrm{sdim}\ I(\Lambda) = -\mathrm{sdim}\ V(\Lambda)_{\mathrm{IR}} = 2.$$

Some exceptions ($a_2 = 0$) and the atypical case $a_1 = 0$ are discussed in Farmer and Jarvis (1983).

2.6. *Exceptional* LSA

These are also type-2 LSA. Thus their MR are built in the same way as $B(m, n)$ or $D(m, n)$. In particular, we now have

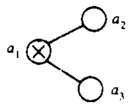
$$D(2, 1, \alpha) \quad T = \frac{1}{1+\alpha}(2h_1 - h_2 - \alpha h_3) \qquad (2.29)$$

$$G(3) \qquad T=\tfrac{1}{2}(h_1-2h_2-3h_3) \tag{2.30}$$

$$F(4) \qquad T=\tfrac{1}{3}(2h_1-3h_2-4h_3-2h_4). \tag{2.31}$$

Thus T is the same time the su(2) Cartan subalgebra generator.

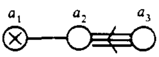
When $b \geqslant \dim G_{-1}$, $\mathrm{Ind}_P^G\, V_0 \approx V(\Lambda)_{\mathrm{MR}}$ coincides with $\bar{V}(\Lambda)$ where

(i) $\bar{V}\Big($  $;\ b=\dfrac{1}{1+\alpha}(2a_1-a_2-\alpha a_3)\Big)=\{(b+1,1,1)+(b,2,2)$

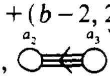
$+(b-1,3,1)+(b-1,1,3)+(b-2,2,2)+(b-3,1,1)\}$

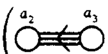
$\otimes(1,a_2+1,a_3+1)$ of su(2)+su(2)+su(2). (2.32)

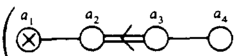
Thus $\dim V(\Lambda)_{\mathrm{MR}}=2^4(b-1)(a_2+1)(a_3+1)$.

(ii) $\bar{V}\Big($  $;\ b=\tfrac{1}{2}(a_1-2a_2-3a_3)\Big)=(\{(b+1,1)+(b,7)$

$+(b-1,14+7))+(b-2,27+7+1)+(b-4,14+7)+(b-5,7)$

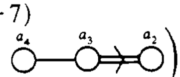
$+(b-6,1)\}\otimes\Big(1,$  $\Big)$ of su(2) + G_2. (2.33)

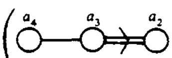
Thus $\dim V(\Lambda)_{\mathrm{MR}}=2^7\{(b-7/2)+1)\}\dim\Big($  $\Big)$.

(iii) $\bar{V}\Big($  $;\ b=\tfrac{1}{3}(2a_1-3a_2-4a_3-2a_4)\Big)=\{(b+1,1)+(b,8)$

$+(b-1,21+7)+(b-2,48+8)+(b-3,35+27+7+1)$

$+(b-4,48+8)+(b-5,21+7)$

$+(b-6,8)+(b-7,1)\}\otimes\Big(1,$  $\Big)$ of su(2) + so(7). (2.34)

Thus $\dim V(\Lambda)_{\mathrm{MR}}=2^8\{(b-9/2)+1\}\cdot\dim\Big($  $\Big)$.

All these dimension formulae correspond in fact exactly to those of the typical IR. Moreover, the superdimension of all these MR are always vanishing, in agreement with the superdimension formulae (Kac 1978) for the typical IR.

3. Young supertableaux

3.1. *Generalisation of Fischler's* YT

We consider here YST corresponding to the Kac-Dynkin diagrams for su(m/n) and osp$(M/2n)$ representations (in general, they will correspond to the HW-MR). They were established by trial and error, the main justification of their respective shapes is that they satisfy the requirements presented in the introduction, as will be proved in the next section. If these YST are similar or essentially equivalent to YST previously defined in earlier works (see Balantekin and Bars (1981), Bars *et al* (1983) and Dondi and Jarvis (1981) for su(m/n) and Farmer and Jarvis (1984) and Morel *et al* (1985) for osp$(M/2n)$), this is certainly not a coincidence. In particular, the correspondence between the su(m/n)-YST of this paper and those of Bars *et al* (1983) is precisely stated in § 8. The YST describing the remaining non-MR are discussed in § 6.

Instead of generalising to the LSA case the Girardi–Sciarino and Sorba YT (some of them have 'negative boxes' (Girardi *et al* 1982, 1983)) or the strange-looking King YT (King and El-Sharkaway 1983) we decide to generalise the Fischler YT (Fischler 1981). The differences among them are significant, especially for the spinor representations, but since we are mainly interested in the 'notation of the highest weight' interpretation of the YST (HW-YST) as opposed to the 'tensor' interpretation (T-YST), and that all the finite spinor representations are typical, the generalisation of the Fischler YT is sufficient for our purpose.

According to this work, the YT of su(N), so(N) and sp(N) have at the most N rows and are almost completely characterised by the IR since

(i) columns with n boxes can be removed or added to give an equivalent su(n)-YT;

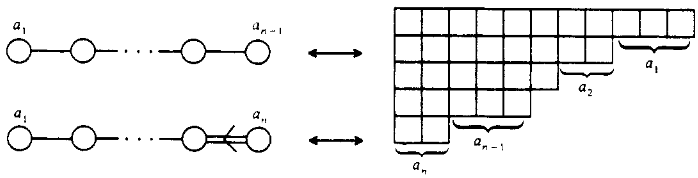
(ii) so(M)-YT with more than $[M/2]$ rows is equivalent to so(M)-YT having at the most $[M/2]$ rows, since there exists a tensor $\mathscr{E}$ which changes columns made of k boxes into columns made of $M-k$ boxes;

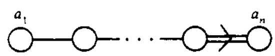
(iii) the same is true for sp(N)-YT with the help of a tensor Ω which is a sp(N) invariant (Fischler 1981).

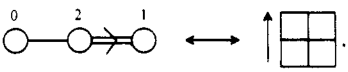
Consequently, there is a unique YT completely characterised by the Dynkin indices where the columns have $k \leqslant R$ boxes (R = rank of the algebra) each.

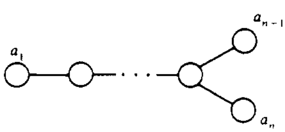
Let c_i be the number of columns with i boxes of a YT, then Fischler's conventions are as follows.

(i) For su($n+1$) and sp($2n$): $c_i = a_i$, $\forall i$.



(ii) For so($2n+1$)  the rules are $c_i = a_i$, $\forall i \leqslant n-1$ and $c_n = [a_n/2]$. If a_n is odd we have to add an arrow $\uparrow$ in front of the YT, i.e.



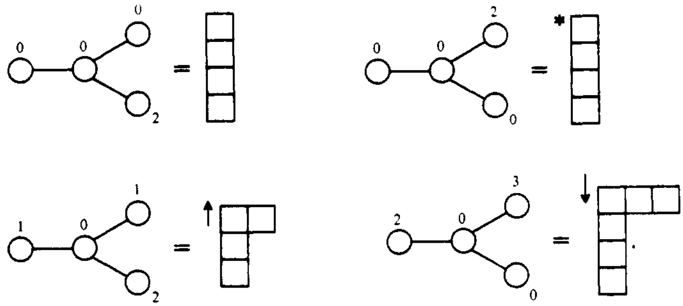
(iii) For so($2n$),  the rules are more complicated:

$$c_i = a_i \qquad i \leqslant n-2 \qquad c_{n-1} = \min(a_{n-1}, a_n) \qquad c_n = [|(a_{n-1} - a_n)|/2].$$

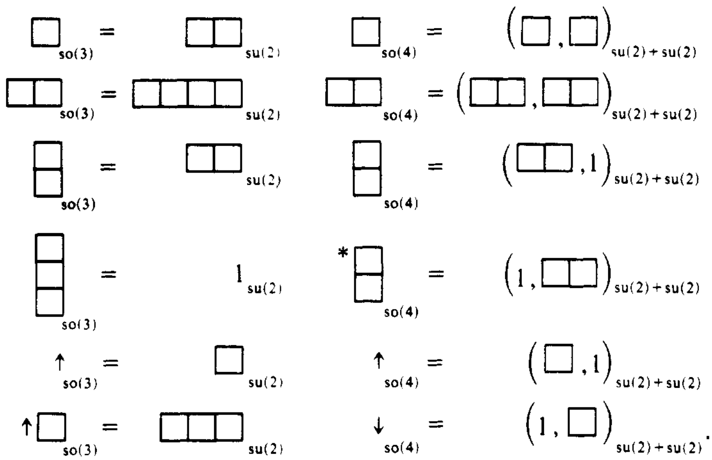
Furthermore we have to add

a star *	if $a_n < a_{n-1}$	$a_{n-1} - a_n = 2k$
an arrow $\uparrow$	if $a_{n-1} \leqslant a_n$	$a_{n-1} - a_n = 2k+1$
an arrow $\downarrow$	if $a_{n-1} > a_n$	$a_{n-1} - a_n = 2k+1$.

Examples are



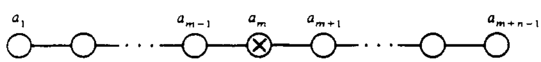
Because of the isomorphisms so(3)$\approx$su(2) and so(4)$\approx$su(2)+su(2) we obtain the following equivalences:

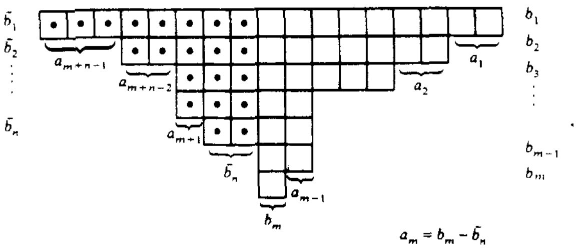


In Abramsky and King (1970) and King (1970) mixed YT (made both of dotted and undotted boxes) were explicitly introduced for the description of u(M) and su(M) IR, and subsequently generalised for u(m/n) and su(m/n) in Dondi and Jarvis (1981) and King (1982). In particular, the similarity of the rules for the respective branching of u($m+n$) versus the branching of u(m/n) into u(m)+u(n) is striking. Although in the su($m+n$) case these mixed YT are equivalent to the conventional ones, their use allows easy rules for the branching into su(m)+su(n)+u(1) (Bars *et al* 1983). On the other hand, the necessity of the use of mixed YST in the u(m/n) and su(m/n) cases is clear. Their adjoint representations themselves cannot be described by any conventional YST.

3.2. Type-1 LSA

For both su(m/n) and osp($2/2n$) we will first take into consideration the case $a_s \in Z$.

For su(m/n)  we propose the following YST:



The eigenvalue of the u(1) generator being

$$q = \frac{1}{n-m}[n \cdot (\#\text{ covariant boxes } \square) - m \cdot (\#\text{ contravariant boxes } \boxdot)].$$

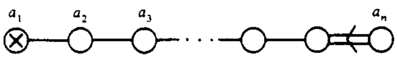
Clearly, to simultaneously add k columns of m covariant indices and k columns of n contravariant indices does not change the Dynkin indices.

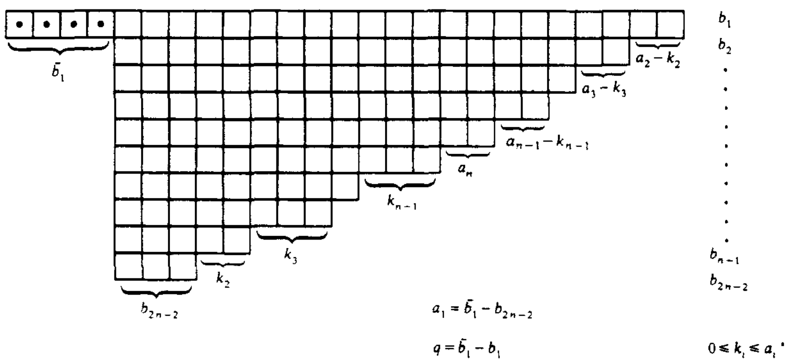
Note that the following non-typicality condition:

$$\sum_{j=i}^{m-1} (a_t + 1) + a_m - \sum_{t=m+1}^{j} (a_t + 1) = 0$$

is in the YST:

$$b_i + (m-i) = \bar{b}_j + (n-j).$$

For osp(2/2n − 2)  we propose the following YST:



Evidently, many YST describe the same HW. Nevertheless, in any case the following non-typicality conditions can be read from the YST according to the following:

$$a_1 = \sum_{t=2}^{k} (a_t + 1) \qquad \bar{b}_1 - b_1 = b_{2n-k-1} - b_k + k - 1 \qquad 1 \leqslant k \leqslant n-1 (*)$$

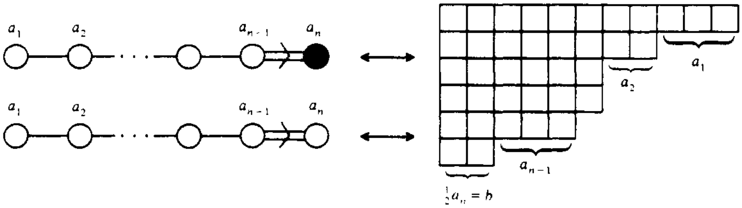
$$a_1 = \sum_{t=2}^{j} (a_t + 1) + 2\sum_{j+1}^{n} (a_t + 1) \qquad \bar{b}_1 - b_1 = b_j - b_{2n-j-1} + 2n - 2 - j \qquad 1 \leqslant j \leqslant n-1$$

but (*) is equivalent to $\bar{b}_1 - b_1 = b_j - b_{2n-j-1} + 2n - 2 - j$, $n \leqslant j \leqslant 2n-2$, thus giving a unified characterisation of the non-typical YST when j varies between 1 and $2n-2$.

When a_s is a non-positive complex number, we consider $\mathrm{I} a_s \mathrm{I} \equiv [\mathrm{Re}(a_s)]$, the integer part of the real part of a_s, for the YST, and we place the number a_s-$\mathrm{I} a_s \mathrm{I}$ in front of the YST in order to give additional information which cannot be obtained by manipulating tensorial indices, the number playing the same role as the arrow for the spinorial YT.

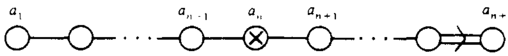
3.3. *Type-2* LSA $osp(M/2n)$

In the case $M = 1$, according to § 2.4, it seems logical to identify the YST shape of an $osp(1/2n)$-IR with the YT shape of the corresponding $so(2n+1)$-IR:

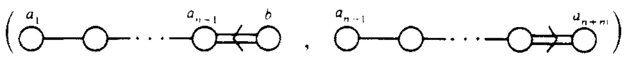


When $M \geqslant 3$, the virtue of the following YST is that they automatically incorporate the consistency conditions.

For $osp(2m+1/2n)$

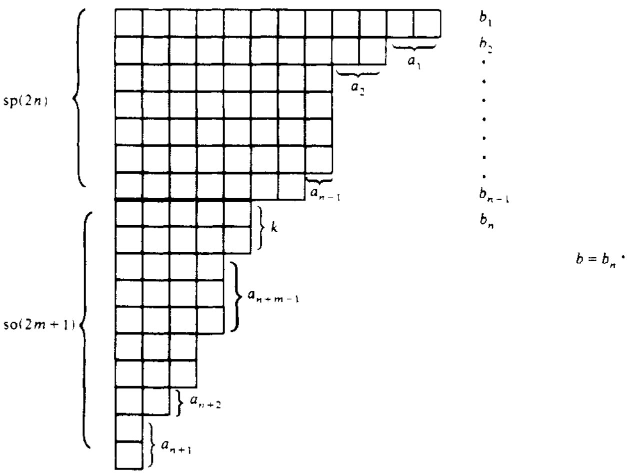


or



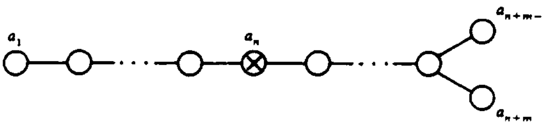
where $b = a_n - a_{n+1} - \ldots - a_{n+m-1} - \frac{1}{2}a_{n+m}$.

(*a*) *Tensor* YST ($a_{m+n} = 2k$)

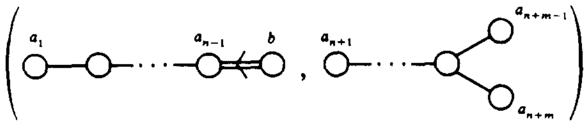


(*b*) *Spinor* YST. When $a_{m+n} = 2k+1$ we have to add in front of the previous YST an arrow, as in the B_n case.

For $D(m, n)$



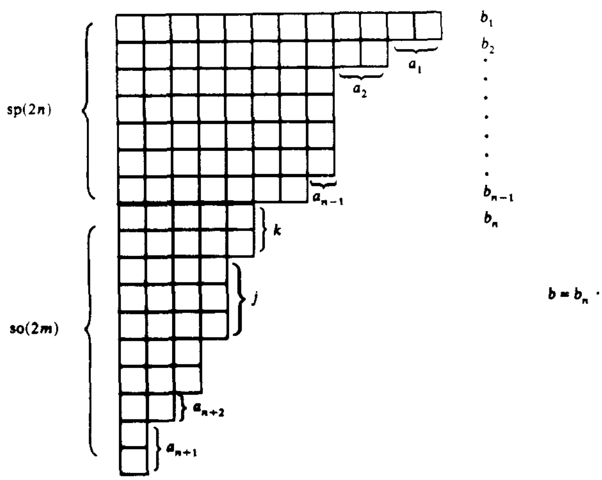
or



where $b = a_n - a_{n+1} - \ldots - a_{n+m-2} - \frac{1}{2}(a_{n+m-1} + a_{n+m})$.

(*a*) *Vector* YST $(a_{n+m-1} + a_{n+m} = 2k)$.

(i) When $a_{n+m-1} \leqslant a_{n+m}$, the following YST should be considered with $j = a_{n+m-1}$ and $k = \frac{1}{2}(a_{n+m} - a_{n+m-1})$:



(ii) When $a_{n+m-1} \geqslant a_{n+m}$, take $j = a_{n+m}$, $k = \frac{1}{2}(a_{n+m-1} - a_{n+m})$ and add a star * in front of the YST, as in the D_n case.

(*b*) *Spinor* YST $(a_{n+m-1} + a_{n+m} = 2k+1)$.

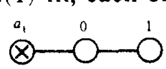
(i) When $a_{n+m-1} \leqslant a_{n+m}$, consider the above YST with $j = a_{n+m-1}$, $k = \frac{1}{2}(a_{n+m} - a_{n+m-1} - 1)$ and add the arrow ↑.

(ii) When $a_{n+m-1} > a_{n+m}$, the procedure is similar and we have to consider the above YST with $j = a_{n+m-1}$, $k = \frac{1}{2}(a_{n+m-1} - a_{n+m} - 1)$ and add the arrow ↓.

4. Identification of the invariant subspaces

We present here three methods for singling out the invariant subspaces of a MR.

The first method consists in the explicit realisation of the expressions for the vectors $|\Gamma\rangle$ where the weights Γ are the highest with respect to the even subalgebra and determine under which conditions these Γ are also HW of the whole superalgebra. In practice the realisation of the so-called 'lower highest weight' as in Hurni and Morel (1982, 1983) is so tedious that it justifies the search for more attractive methods. Therefore we just give an example for su(1/3) (low-dimensional osp(M/N) are treated in that way in Farmer and Jarvis (1984)).

According to § 2, a su(1/3)-MR is made of several su(3)+u(1)-IR, each one being specified by a su(3)+u(1)-HW that we shall call Γ_i. For example, 

consists

of

$$3^*_q+(8+1)_{q-1}+(6^*+3)_{q-2}+3^*_{q-3}\equiv\sum_{\{i\}}V_0(\Gamma_i)$$

$$q=\tfrac{1}{2}(3a_1-1).$$

The even roots of su(1/3) are in $\Delta_0=\{\pm(\varepsilon_i-\varepsilon_j)\}$, $i\neq j=1,2,3$, and the odd ones in $\Delta_1=\{\pm(\varepsilon_0-\varepsilon_i)\}$, $i=1,2,3$. The generators $b_i^+\equiv E_{+(\varepsilon_0-\varepsilon_i)}$ and $b_i\equiv E_{-(\varepsilon_0-\varepsilon_i)}$ are in a 3^*_{+1} (respectively 3_{-1}) of su(3)+u(1). For convenience we will also denote the $E_{+\alpha}$ by α^+ and the $E_{-\alpha}$ simply by α.

By definition of the G_0-HW Γ_i we should have $\alpha_2^+|\Gamma_i\rangle=\alpha_3^+|\Gamma_i\rangle=0\ \forall\Gamma_i$. If $b_1^+|\Gamma_i\rangle\neq 0$ then Γ_i is not a highest weight of su(1/3).

When $b_1^+|\Gamma_i\rangle=0$, as is explained in § 2, we have two possibilities: either Γ_i is the highest weight of an invariant subspace or this presumed invariant subspace is in fact completely decoupled.

Direct calculations for each $V_0(\Gamma_i)$ give the following table:

Given $V_0(\Gamma_i)$	$\vert\Gamma_i\rangle$ is	and $b_1^+\vert\Gamma\rangle$ is equal to
3^*_q	$\vert\Lambda\rangle$	0 by definition of $\vert\Lambda\rangle$
8_{q-1}	$b_1\vert\Lambda\rangle$	$a_1\vert\Lambda\rangle$
1_{q-1}	$(3b_1\alpha_2\alpha_3-2\alpha_2b_1\alpha_3+\alpha_3\alpha_2b_1)\vert\Lambda\rangle$	$(a_1-3)\alpha_2\alpha_3\vert\Lambda\rangle$
6^*_{q-2}	$b_1\alpha_2b_1\vert\Lambda\rangle$	$(a_1-1)\alpha_2b_1\vert\Lambda\rangle$
3_{q-2}	$(\alpha_3b_1\alpha_2b_1-2b_1\alpha_2b_1\alpha_3)\vert\Lambda\rangle$	$(a_1-1)\vert\Gamma(1_{q-1})\rangle-(a_1-3)\underbrace{b_1\alpha_2\alpha_3\vert\Lambda\rangle}_{\in 8_{q-1}}$
3^*_{q-3}	$b_1\alpha_2\alpha_3b_1\alpha_2b_1\vert\Lambda\rangle$	$(a_1-2)\alpha_2\alpha_3\vert\Gamma(6^*_{q-2})\rangle-(a_1-1)b_1\alpha_2\alpha_3\alpha_2b_1\vert\Lambda\rangle$

Thus, as we should, we recover the non-typicality conditions $a_1=0$, $a_1=a_2+1=1$ and $a_1=a_2+a_3+2=3$.

Consulting the above table, we see that $\Gamma(3^*_{q-3})$ can never be the HW of an invariant subspace; fortunately since the only 'trivially graded' representation is the one-dimensional one. $\Gamma(3_{q-2})$ cannot be a candidate since we go back to the 8_{q-1} (respectively the 1_{q-1}), when $a_1=1$ (respectively 3).

Then we have

$$\overset{0}{\otimes}\!-\!\!-\!\overset{0}{\bigcirc}\!-\!\!-\!\overset{1}{\bigcirc}=(3^*_{-1/2}+1_{-3/2})\text{ or }(3^*_{-1/2}+1_{-3/2})\oplus(8_{-3/2}+6^*_{-5/2}+3_{-5/2}+3^*_{-7/2})$$

$$\overset{1}{\otimes}\!-\!\!-\!\overset{0}{\bigcirc}\!-\!\!-\!\overset{1}{\bigcirc}=(3^*_1+8_0+1_0+3_{-1})\text{ or }(3^*_1+8_0+1_0+3_{-1})\oplus(6^*_{-1}+3^*_{-2})$$

$$\overset{3}{\otimes}\!-\!\!-\!\overset{0}{\bigcirc}\!-\!\!-\!\overset{1}{\bigcirc}=(3^*_4+8_3+6^*_2)\text{ or }(3^*_4+8_3+6^*_2)\oplus(1_3+3_2+3^*_1).$$

In particular, note that in each invariant subspace

$$\sum_{\text{even }V_0(\Gamma_i)}q(i)\cdot\dim V_0(\Gamma_i)=\sum_{\text{odd }V_0(\Gamma_i)}q(i)\cdot\dim V_0(\Gamma_i).$$

This leads to the second method.

Let $q(i)$ be the u(1) eigenvalue associated to each $V_0(\Gamma_i)$, then the supertrace of the operator Q in this representation is given by

$$\mathrm{STr}\, Q = \sum_{\{i\}} (-1)^{(q-q(i))} q(i) \cdot \dim V_0(\Gamma_i).$$

As any invariant subspace of $V(\Lambda)_{\mathrm{MR}}$ should satisfy $\mathrm{STr}\, Q = 0$ it is easy to verify if a space built from a given $|\Gamma_i\rangle$ can be a candidate for such a subspace. In our former example, we have the following.

(i) From 1_{q-1} we can envisage building the subspace $I(\Lambda) = 1_{q-1} + 3_{q-2} + 3^*_{q-3}$ (since $6^*_{q-2} \not\subset 1_{q-1} \otimes 3_{-1}$), but $\mathrm{STr}\, Q = 0$ iff $q = 4$, i.e. $a_1 = 3$. We already know that in this case $I(\Lambda)$ is the good one.

(ii) $I(\Lambda) = 3_{q-2} + 3^*_{q-3}$ can never have $\mathrm{STr}\, Q = 0$.

(iii) $I(\Lambda) = 3^*_{q-3}$ has $\mathrm{STr}\, Q = 0$ for $q = 3$, i.e. $a_1 = \frac{7}{3}$, but it is not a non-typicality condition.

(iv) $I(\Lambda) = 1_{q-1}$ has $\mathrm{STr}\, Q = 0$ for $q = 1$, i.e. $a_1 = 1$, a true non-typicality condition, and furthermore 1_0 is a true su(1/3) representation. However, we know from the first method that in fact this presumed $I(\Lambda)$ is not an invariant subspace. For this reason we must be careful about the second method which derives from a necessary but insufficient condition.

Remark. In the case of su(n/n) we have

$$\mathrm{STr}\, C = \sum_{\{i\}} (-1)^{\deg V_0(\Gamma_i)} c \cdot \dim V_0(\Gamma_i) = c \cdot \mathrm{sdim}\, I(\Lambda)$$

c being constant for the whole $V(\Lambda)_{\mathrm{MR}}$. Since we must have $\mathrm{STr}\, C = 0$, this implies $\mathrm{sdim}\, V(\Lambda)_{\mathrm{IR}} = \mathrm{sdim}\, I(\Lambda) = 0$ when $c \neq 0$.

The third (YST) method comes from the observation that, if the non-typicality condition $(\Lambda + \rho, \alpha) = 0$ holds for a given α, then $\chi = \Lambda - \alpha$ is the highest weight of $I(\Lambda)$ if the vector $|\chi\rangle$ really exists in $V_0(\Lambda) \otimes G_{-1}$. The comparison of the respective YST of $V(\Lambda)$ and $I(\Lambda)$ is in fact meaningful.

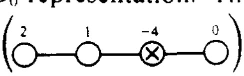
(i) su(m/n). Let an HW be such that the non-typicality condition

$$(\Lambda + \rho, \alpha) = 0 = \sum_{t=i}^{m-1} (a_t + 1) + a_m - \sum_{t=m+1}^{j} (a_t + 1)$$

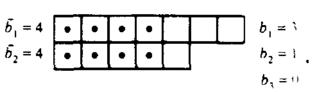
holds. Then we already know the corresponding YST satisfies $b_i - i + m = \bar{b}_j - j + n$. Finally, when we compute the YST corresponding to $\chi = \Lambda - \alpha$, we see that we can obtain it from the former by simply removing a covariant box from the b_ith row and a contravariant one from the $\bar{b}_j$th row.

Intuitively this consists in contracting these indices in order to single out a YST which characterises $I(\Lambda)$.

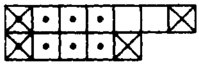
According to that point of view, one may suspect that, when $V_0(\Lambda - \alpha) \not\subset V_0(\Lambda) \otimes G_{-1}$, we are obliged to remove more than one pair of boxes in order to get a legal YST characterising this G_0 representation. This is indeed the case.

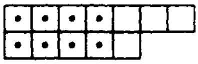
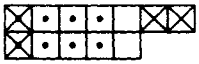
Example 1. Let V 

, then we have the su(3/2) non-typicality condition $a_1 + a_2 + a_3 - a_4 + 1 = 0$. The simplest corresponding YST is



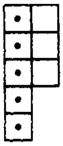
.

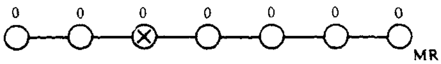
Now (i) tells us that we have to remove a box in the b_1st row and another one in the $\bar{b}_1$st row, but this last one cannot be removed without the first box of the $\bar{b}_2$nd row in order to have a well behaved YST. Thus we have to remove a second covariant box. By using the second method we find that the following YST , where the crossed boxes denote deleted boxes, cannot describe an invariant subspace of $V(\Lambda)$. In fact, the removal of two boxes in the b_1st row, which is the only possible solution permitting the removal of no more than two pairs of boxes, is the correct solution:

$V(\Lambda) =$  $\supset$  $= I(\Lambda)$.

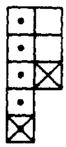
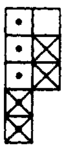
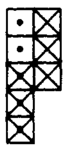
Remark. Being obliged to contract more than one pair of indices means that, in spite of the non-typicality of $V(\Lambda)$, we have $E_{-\alpha}|\Lambda\rangle \neq 0, \forall \alpha \in \Delta_1$. This confirms the assertion (2.4′).

If the resulting (legal) YST does not describe any possible $I(\Lambda)$, we again have to remove other boxes from the b_ith and $\bar{b}_j$th rows considered above.

Example 2. In su(3/5)  seems to be able to describe



since $b_3 + (m-3) = \bar{b}_5 + (n-5)$, $b_2 + (m-2) = \bar{b}_4 + (n-4)$ and $b_1 + (m-1) = \bar{b}_3 + (n-3)$.

But we cannot describe the respective $I(\Lambda)$ by , ,  which have to imply the following $I(\Lambda)_0$: the $(3^*, 5)_{-1}$, $(3, 10)_{-2}$ and $(1, 10')_{-3}$, respectively. However, only the $(3^*, 5)_{-1}$ really figures in

$$V(\Lambda)_{\mathrm{MR}} = \{(1,1)_0 + (3^*,5)_{-1} + (6^*,10)_{-2} + (3,15)_{-2} + (1,35)_{-3} + \ldots + (1,1)_{-15}\}.$$

Such is not the case for the $(3, 10)_{-2}$ and $(1, 10')_{-3}$.

According to Hurni and Morel (1983), the HW Γ_i of the invariant subspaces V_i are in fact those of the $(3^*, 5)_{-1}$, $(6, 50)_{-4}$ and $(1, 175)_{-9}$, where more precisely we have

$$|\Gamma_1\rangle \equiv |\Lambda\rangle$$

$$|\Gamma_2\rangle = b_3^3|\Gamma_1\rangle$$

$$|\Gamma_3\rangle = b_4^3 \cdot b_4^2 \cdot b_3^2|\Gamma_2\rangle$$

$$|\Gamma_4\rangle = b_5^3 \cdot b_5^2 \cdot b_5^1 \cdot b_4^1 \cdot b_3^1|\Gamma_3\rangle$$

where the b_j^i denote the generators corresponding to the roots $\alpha = -(\alpha_i + \alpha_{i+1} + \ldots + \alpha_j)$.

As $|\Gamma_i\rangle = 0$ implies $|\Gamma_j\rangle = 0$ for $j > i$, we have

$$V(\Lambda)_{\mathrm{MR}} = V_1 \oplus (V_2 \oplus (V_3 \oplus V_4)).$$

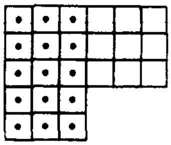
By factorisation of the invariant subspaces, the other possible indecomposable representations are

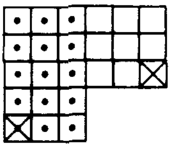
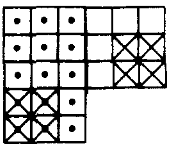
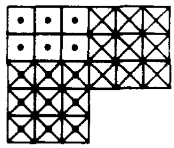
$$V(\Lambda)' = V_1 \oplus (V_2 \oplus V_3)$$

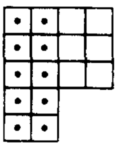
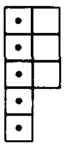
$$V(\Lambda)'' = V_1 \oplus V_2$$

$$V(\Lambda)''' = V_1, \text{ the trivial one-dimensional IR.}$$

In particular, it is impossible to have $V_1 \oplus V_3$ or any other combination of irreducible subspace.

Consequently, we have to take at least  in order to describe $V(\Lambda)_{\mathrm{MR}}$; the highest weight of all the invariant subspaces being characterised, respectively, by the YST obtained by simply deleting the crossed boxes mentioned:

,  and .

For the same reason, ,  and 1 describe at best $V(\Lambda)'$, $V(\Lambda)''$ and $V(\Lambda)'''$, respectively.

These two examples provide an illustration of the (heuristic) third method.

(i) Let $V(\Lambda)$ be described by a YST such that $b_i - i + m = \bar{b}_j - j + n$. Then we get the YST characterising the invariant subspace $I(\Lambda)$ by deleting one covariant box from the b_ith row and another contravariant one in the $\bar{b}_j$th row.

(ii) If the resulting YST is illegal, then remove enough boxes in order to get a legal YST.

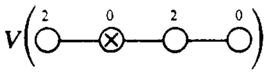
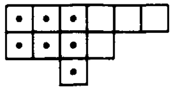
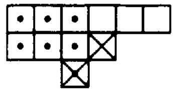
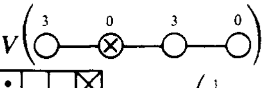
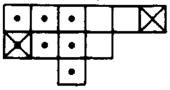
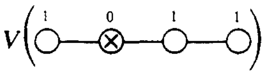
(iii) This is done by taking care to remove each set of symmetrised covariant boxes together with the corresponding antisymmetrised contravariant boxes, and vice versa, as shown in the first example of appendix 2.

In particular, (iii), together with (ii), explain quickly why we need to remove four or nine pairs of boxes, respectively, in the previous example.

Using the Littlewood–Richardson rule, it is easy to verify, if we really have the expected $I_0(\Lambda)$ in $V_0(\Lambda) \otimes \Lambda^k G_{-1}$, where k is the number of pairs of boxes removed from the YST of $V(\Lambda)_{\mathrm{MR}}$.

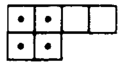
The first YST describes by definition an IR if and only if the algorithm does not produce an invariant subspace.

When many non-typicality conditions overlap as in example 2, the irreducible invariant subspace characterised by a YST involving a determined number of crossed boxes cannot be factorised from $V(\Lambda)_{\mathrm{MR}}$ without all the irreducible invariant subspaces characterised by YST involving a larger number of boxes. When the non-typicality conditions do not overlap, as in the following example, the irreducible invariant subspaces may be factorised independently.

Example 3. Let V$_{\mathrm{MR}}$ have a YST of  ($b_2 = \bar{b}_3$ and $b_1 + 1 = \bar{b}_2 + 2$). Therefore, the two invariant subspaces characterised by $(\Lambda+\rho, \alpha_2)$ and $(\Lambda+\rho, \alpha_1+\alpha_2+\alpha_3)$ do not overlap and we have $V(\Lambda)_{\mathrm{MR}} = V_2 \oplus V_1 \oplus V_3$, where the irreducible invariant subspaces are $V_2 = V(\Lambda-\alpha_2)_{\mathrm{IR}}$, characterised by , i.e. V$_{\mathrm{IR}}$, and $V_3 = V(\Lambda-\alpha_1-\alpha_2-\alpha_3)_{\mathrm{IR}}$, characterised by , i.e. V$_{\mathrm{IR}}$.

The possible indecomposable representations are then

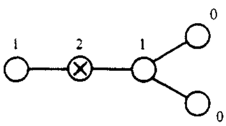
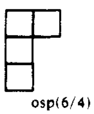
$$V(\Lambda)_{\mathrm{MR}} = V_2 \oplus V_1 \oplus V_3 \qquad V(\Lambda)' = V_1 \oplus V_2$$
$$V(\Lambda)'' = V_1 \oplus V_3 \qquad V(\Lambda)''' = V_1 = V(\Lambda)_{\mathrm{IR}}.$$

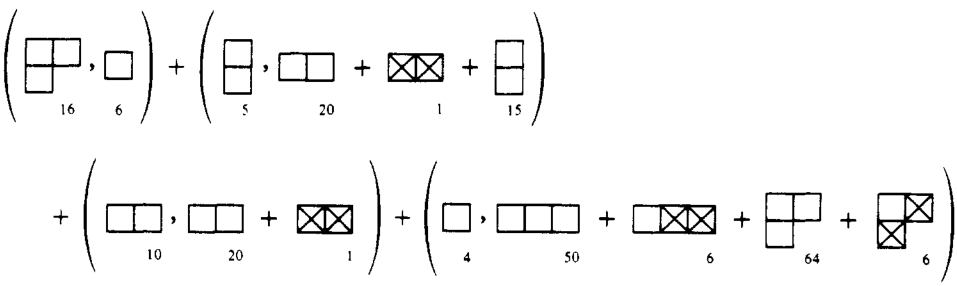
If $V(\Lambda)'$ may also be described by a simple YST, namely , this is not the case for $V(\Lambda)''$ and $V(\Lambda)'''$. Specific YST able to characterise these kinds of non-MR are defined in § 6.

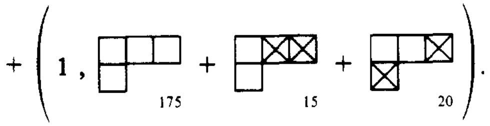
(ii) osp$(2/N)$. Let the non-typicality condition $\bar{b}_1 - b_1 = b_1 - b_{N+1-j} + N - j$. Then we have to contract one box of the $\bar{b}_1$st row with one of the b_jth row, when $2 \leq j \leq N$. When $j = 1$, we have to contract one box of the $\bar{b}_1$st row with one of the b_Nth row, plus two boxes of the $\bar{b}_1$st row with two others of the b_1st row.

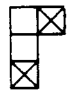
When some $b_j = b_{j+1}$, there can be problems that we can master with the second method as in appendix 2.

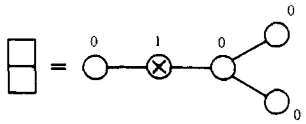
(iii) osp(M/N), $M \geq 3$. The third method is easily adapted for these type-2 LSA.

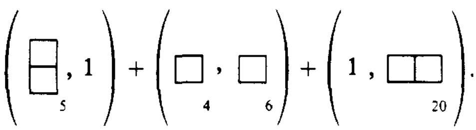
When we have $b_i - (N-i) = c_j - (M-1-j)$ then we have to contract a 'sp(N)' box in the b_ith row and a 'so(M)' one in the c_jth column. For example, the YST corresponding to  is , the MR is made of

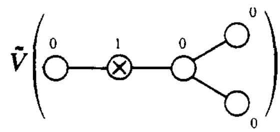


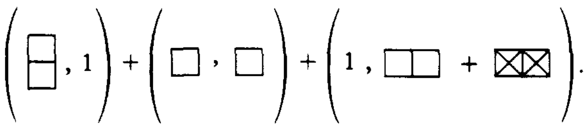


The non-typicality condition $b_1-(4-1)=c_1-(6-1-1)$ can be written  . Then

$I(\Lambda)$ specified by  is made of



Remark. In § 2 we considered  made of

 . If $\tilde{V}(\Lambda)$ was $V(\Lambda)_{\mathrm{MR}}$ where $V(\Lambda)_{\mathrm{MR}}=V(\Lambda)_{\mathrm{IR}}\oplus 1$, the singlet would have been obtained by contraction of the $V(\Lambda)_{\mathrm{MR}}$ indices, both of the 'sp(N)' type. It is an irrelevant possibility according to the third method. Thus $\tilde{V}(\Lambda)=V(\Lambda)_{\mathrm{IR}}\oplus 1$ implies $V(\Lambda)_{\mathrm{MR}}=V(\Lambda)_{\mathrm{IR}}$ in this case.

5. Pseudo-highest weight representations

The pseudo-highest-weight (or generalised highest weight) representations do appear necesary when studying the unitarisable (infinite) integral representations of the non-compact semisimple Lie algebra.

Let $G=K+N$ be a simple real LA, where K is its maximal compact Lie subalgebra. If Δ_k, Δ_n denote the set of compact (non-compact roots respectively) we define $\tilde{\Delta}=\Delta_k\cup(-\Delta_n)$.

Let μ denote a highest weight relative to $\tilde{\Delta}$, i.e. $E_{+\alpha}|\mu\rangle=0$ $\forall\alpha\in\tilde{\Delta}$. According to Williams (1982), Parthasarathy has given a necessary and sufficient condition on μ for the unitarisability of $V(\tilde{\Delta},\mu)$ when μ is regular, i.e. $(\mu+\rho_k-\rho_n,\alpha)\neq 0$ $\forall\alpha$, where $\rho_k,\rho_n=\frac{1}{2}\Sigma\alpha$, $\alpha\in\Delta_k^+$ (respectively Δ_n^+):

$$2(\mu,\alpha)/(\alpha,\alpha)\in Z_+\cup\{0\},\ Z,\ \forall\alpha\in\Delta_k\ (\Delta_n \text{ respectively}). \tag{5.1}$$

In the language of superalgebras, such representations would be called typical. Williams (1982) has corroborated the validity of (5.1) in the su(2, 2) non-regular case.

Note that Δ and $\tilde{\Delta}$ are related by Weyl reflections. This therefore leads to the following general definition.

Definition. A pseudo-highest weight μ is such a weight that a representation is characterised by a vector $|\mu\rangle$ satisfying $E_{+\alpha}|\mu\rangle \equiv 0\ \forall\alpha \in \tilde{\Delta}$, where the system of pseudo-roots $\tilde{\Delta}$ is obtained from Δ by a particular succession, W, of Weyl reflections: $\tilde{\Delta} = W(\Delta)$.

Significant examples were the representations with lowest weight (LW): $\tilde{\Delta}^+ = -\Delta^+ (\equiv \Delta^-)$. When there are many Weyl-inequivalent systems of simple roots, say $\Delta_{(i)}$, then there are no Weyl reflections W such that $\Delta_{(i)} = W(\Delta_{(j)})$. Therefore, for each $\Delta_{(i)}$, there exist corresponding systems of pseudo-roots: $\tilde{\Delta}_{(i)} \equiv W(\Delta_{(i)})$.

By extension, if we call $\Delta^+_{(1)}$ 'the' positive roots, then the $\Delta^+_{(i)}$, $i \neq 1$, will also be considered 'pseudo-positive' roots (or positive roots with respect to $\Delta_{(i)}$).

Definition. $V(\tilde{\Delta}, \mu)$ denotes a representation characterised by a pseudo-highest weight μ, i.e. a weight which is highest with respect to the pseudo-simple root corresponding to a particular choice of $\tilde{\Delta}$.

In the finite-dimensional case, the property of complete reducibility of the semisimple LA guarantees the existence of isomorphism between finite $V(\Delta; \Lambda)$ and $V(\tilde{\Delta}; \mu)$, as in the following example:

$$\left(\Delta, \overset{0}{\bigcirc}\!-\!\!-\!\overset{1}{\bigcirc}\right) = 3^* = \left(\tilde{\Delta}, \overset{-1}{\bigcirc}\!-\!\!-\!\overset{0}{\bigcirc}\right) \quad \text{when } \tilde{\Delta} = -\Delta$$

since it is well known that for a simple LA the lowest weight of $V(\Lambda)$ is minus the highest weight of $V(\Lambda)^*$:

$$V(\tilde{\Delta}, \Lambda) \approx V(-\tilde{\Delta}, -\Lambda^*). \tag{5.2}$$

For the type-2 LSA case the isomorphism (5.2) remains true for the IR, but for the type-1 LSA, (5.2) has to be changed. For example, in the typical case

$$V(\tilde{\Delta}, \Lambda) \approx V\left(-\tilde{\Delta}, \left(\frac{q}{2N} - 1\right)\rho_1 - \Lambda^*\right) \qquad N = \dim G_{-1}. \tag{5.3}$$

But (5.2) or (5.3), according to the LSA type, is not valid for the MR. For example, consider the LW-MR $\overset{0}{\otimes}\!-\!\!-\!\overset{-1}{\bigcirc}\!-\!\!-\!\overset{0}{\bigcirc}$ whose decomposition into $su(3)+u(1)$ IR is

$$3^*_1 \otimes \{1_0 + G_{+1} + [G_{+1} \cap G_{+1}] + [G_{+1} \cap G_{+1} \cap G_{+1}]\}$$
$$= 3^*_1 \otimes \{1_0 + 3^*_1 + 3_2 + 1_3\} = 3^*_1 + (6^* + 3)_2 + (8+1)_3 + 3^*_4.$$

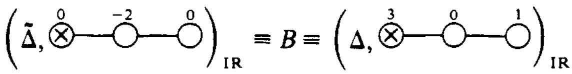
$|\Gamma\rangle \equiv b^+_1|\mu\rangle$ is characterised by the pseudo-HW Γ of the 6^*, but since $a_1 = 0$, any $E_{-\alpha}|\Gamma\rangle = 0$, $\forall\alpha \in \tilde{\Delta}$, then Γ is also the pseudo-HW of the ideal $b^*_2 + 8_3 + 3_4$. Hence, in spite of their identical weights

$$\left(\tilde{\Delta}, \overset{0}{\otimes}\!-\!\!-\!\overset{-1}{\bigcirc}\!-\!\!-\!\overset{0}{\bigcirc}\right)_{\text{MR}} \equiv A \oplus B \neq A \dot{\oplus} B \equiv \left(\Delta, \overset{3}{\otimes}\!-\!\!-\!\overset{0}{\bigcirc}\!-\!\!-\!\overset{1}{\bigcirc}\right)_{\text{MR}}$$

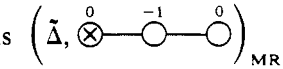
where

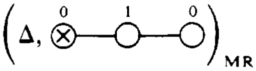
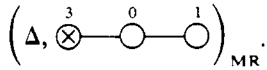
$$\left(\tilde{\Delta}, \overset{0}{\otimes}\!-\!\!-\!\overset{-1}{\bigcirc}\!-\!\!-\!\overset{0}{\bigcirc}\right)_{\text{IR}} \equiv A \equiv \left(\Delta, \overset{2}{\otimes}\!-\!\!-\!\overset{0}{\bigcirc}\!-\!\!-\!\overset{0}{\bigcirc}\right)_{\text{IR}}$$

and



$\tilde{\Delta} = -\Delta.$

Furthermore, it is  which is the SSR conjugate to

 ,not  .

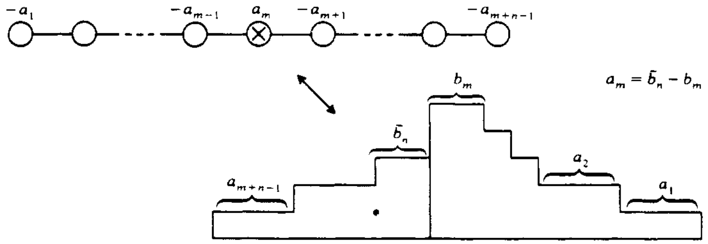
In fact, as for LA, it is clear that for type-1 LSA the conjugate of any PHW-SRR is a PLW one (pseudo-lowest one):

$$V(\tilde{\Delta}, \Lambda^*) \approx V(-\tilde{\Delta}, -\Lambda). \tag{5.4}$$

According to the above example, it is not hard to convince oneself that any PHW representation of su(1/3) is isomorphic to a HW or LW one, since all the SRR built in the manner indicated in this paper have at most one invariant subspace. This is also the case for any of the LSA whose (P)HW representations can never have more than one invariant subspace (i.e. when the grey node is at an extremity of a Dynkin diagram: su$(1/n)$, $C(n)$, $B(1, n)$, $B(m, 1)$, $D(m, 1)$, $D(2, 1, \alpha)$, $F(4)$, $G(3)$).

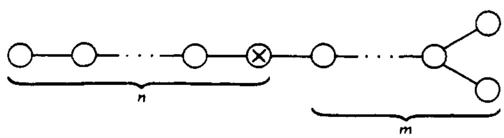
The SRR of the other LSA may have three or more irreducible subspaces. As we have seen in the former example, the choice of the positive roots implies which irreducible subspace is the factor space and which one is the invariant subspace. The generalisation is now obvious: indeed, if a multi-non-typical representation $V(\Delta; \Lambda)$ is isomorphic to $V_1 \oplus V_2 \oplus V_3$, say, then clearly there exists a weight μ such that $V(-\Delta; \mu)$ is isomorphic to $V_1 \oplus V_2 \oplus V_3$, but in addition there are certainly choices of positive roots $\tilde{\Delta}_i$, different from both Δ and $-\Delta$, and different weights Γ_i, such that the corresponding PHW representations $V(\Delta_i; \Gamma_i)$ are isomorphic to $V_1 \oplus V_2 \oplus V_3$, or to $V_1 \oplus V_2 \oplus V_3$, or possibly to other combinations such as $V_1 \oplus V_3 \oplus V_2$.

The YST for the su(m/n)-LW representations may be chosen as

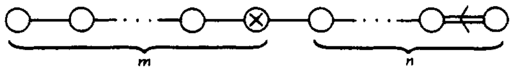


More generally, the (P)LW-YST of any LSA can be defined so that they look like those of the (P)HW-YST, reflected on a horizontal axis.

An example of a PHW representation which is not a HW or LW representation can be found for $G = \text{osp}(2m/2n)$ where $G_{\bar{0}} = \text{so}(2m) + \text{sp}(2n)$ and $G_{\bar{1}} = (2m, 2n)$ as a $G_{\bar{0}}$ IR. By simply considering its two inequivalent systems of simple roots

$D(m, n)$ 

and

$C(m, n)$ 

it is clear that the positive roots of $\tilde{G}_{+1}$, say, in the $D(m, n)$ system of simple roots do coincide neither with G_{+1} nor with G_{-1}.

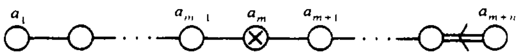
More precisely we have

$\left.\begin{array}{l} G_{+1} = (2m, n) \\ G_{-1} = (2m, \bar{n}) \end{array}\right\}$ according to the so(m)+su(n) subalgebra of $G_{\bar{0}}$ that we get by suppression of the grey node of the $D(m, n)$ Dynkin diagram.

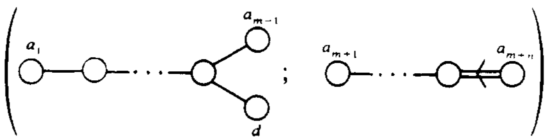
$\left.\begin{array}{l} \tilde{G}_{+1} = (m, 2n) \\ \tilde{G}_{-1} = (\bar{m}, 2n) \end{array}\right\}$ according to the su(m)+sp$(2n)$ subalgebra of $G_{\bar{0}}$ that we get by suppression of the grey node of the $C(m, n)$ Dynkin diagram.

Then a HW-SRR according to $C(m, n)$ is a PHW one according to $D(m, n)$.

It is clear that the YST of a $C(m, n)$ representation characterised by

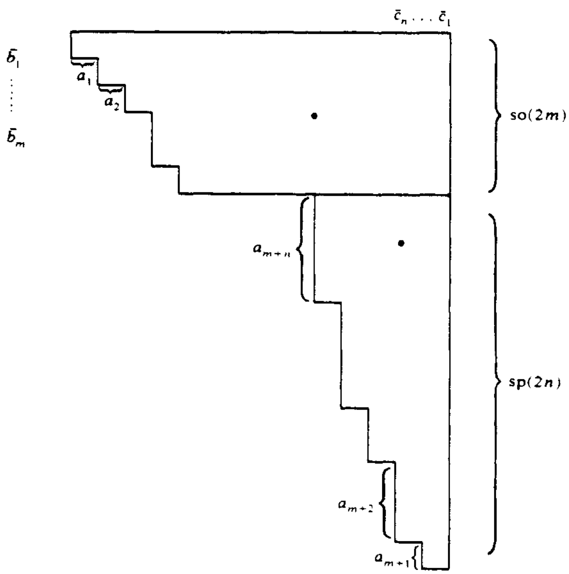


i.e.



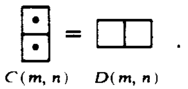
$$d = a_{m-1} + 2a_m - 2 \sum_{t=m+1}^{m+n} a_t$$

can be defined as



$$\bar{b}_m = d$$

Remark. As for the type-1 LSA, only the IR can be defined in any system of simple roots. This is particularly true for the adjoint:



6. Unequivocal characterisation of indecomposable representations by means of YST

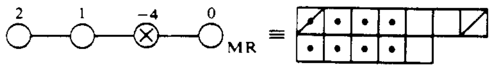
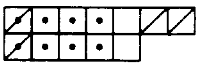
In the previous sections, crossed boxes ⊠ were used in two different ways.

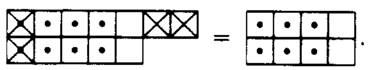
Firstly, to indicate in which manner we have to 'contract the indices' represented by means of Y(S)T in order to single out the invariant subspaces. But in fact, in spite of a legitimate impression, these Y(S)T were not used as a notation for the representations themselves but, on the contrary, were used only as a notation for their respective HW.

Secondly, to 'follow' the action on some weights, described succintly by means of YT, when the odd generators are successively applied to $|\Lambda\rangle$.

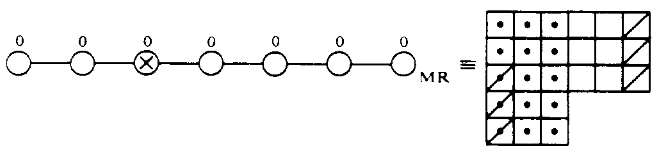
Hence, until now, the crossed Y(S)T were indirectly used to describe some IR for LA and supposed to describe the MR for the the LSA. Now we want to describe the whole representation by means of YST. Therefore precautions are needed since there are inequivalent representations having the same HW (or PHW). This is the reason why we introduce a more complete notation than before for the YST.

6.1. *Definitive notations for the YST*

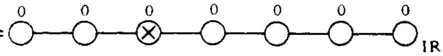
(*a*) Let a YST be defined as in § 3, i.e. it presumably describes a MR. If a non-typicality condition involves the *i*th row and *j*th row in the type-1 case (or *j*th column in the type-2 case), then this condition has to be indicated by a slash (/) in the first box of these rows (or columns), e.g.  as $b_1+(3-1)=b_1+(2-1)$, but not , in spite of the fact that, according to the previous notations, $I(\Lambda)$ is characterised by the following highest weight:

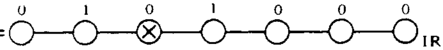


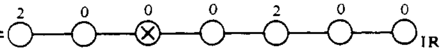
(*b*) If many non-typicality conditions are simultaneously satisfied, then we apply the notation for each condition:

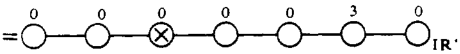


as $b_i+(3-i)=\bar{b}_j+(5-j)$ when $j=i+2$, $i=1,2,3$. We already know that this MR consists of four 'nested' spaces (in the semidirect sum sense): $V_1 \oplus (V_2 \oplus (V_3 \oplus V_4))$, where

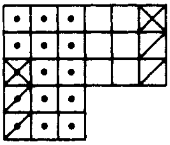
$V_1 =$  $= \mathbf{1}$ of su(3/5)

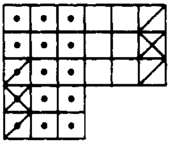
$V_2 =$ 

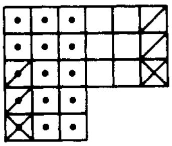
$V_3 =$ 

$V_4 =$ .

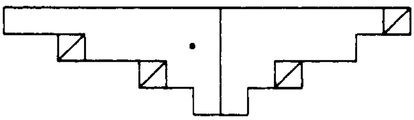
(c) If we are interested in a non-maximal representation, say $V(\Lambda)_{\mathrm{MR}}/I_\alpha(\Lambda)$, where $I_\alpha(\Lambda)$ is the ideal specified by the non-typicality condition $(\Lambda+\rho, \alpha)=0$, then we put a cross ($\times$) instead of the slash (/) in the boxes involved with this non-typicality condition:

 $= V_1 \oplus V_2 \oplus V_3$ when $\alpha = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$

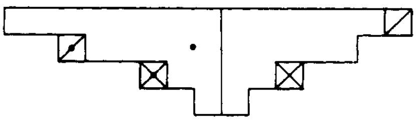
 $= V_1 \oplus V_2$ when $\alpha = \alpha_2+\alpha_3+\alpha_4$

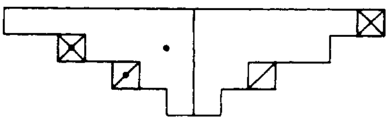
 $= V_1$ when $\alpha = \alpha_3$.

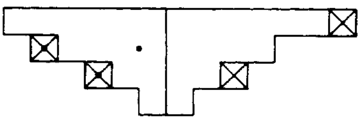
(d) In general, the N invariant subspaces, appearing when the highest weight is N times non-typical, are not all 'nested', as is shown in § 4. Let

 be made of $V_1 \begin{smallmatrix} \oplus V_2 \\ \oplus V_3 \end{smallmatrix}$.

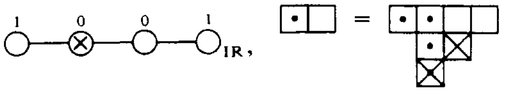
So then

 $\equiv V_1 \oplus V_2$

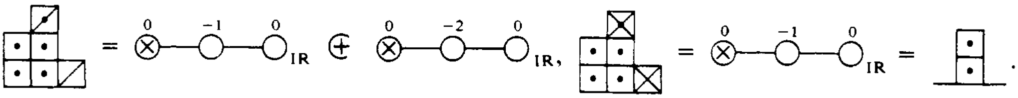
 $\equiv V_1 \oplus V_3$

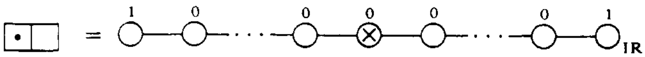
 $\equiv V_1$.

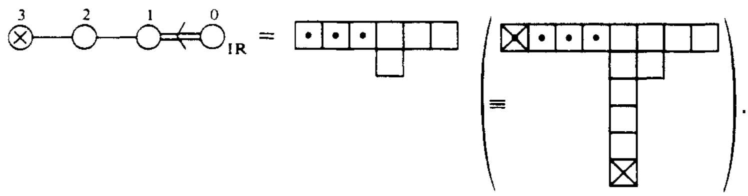
Remark 1. In contrast to the previous YST notations, these new ones allow us to describe unequivocally all the HW representations. But, as in the Lie algebra case, a given representation can still have many equivalent YST. For example, for su(2/3):



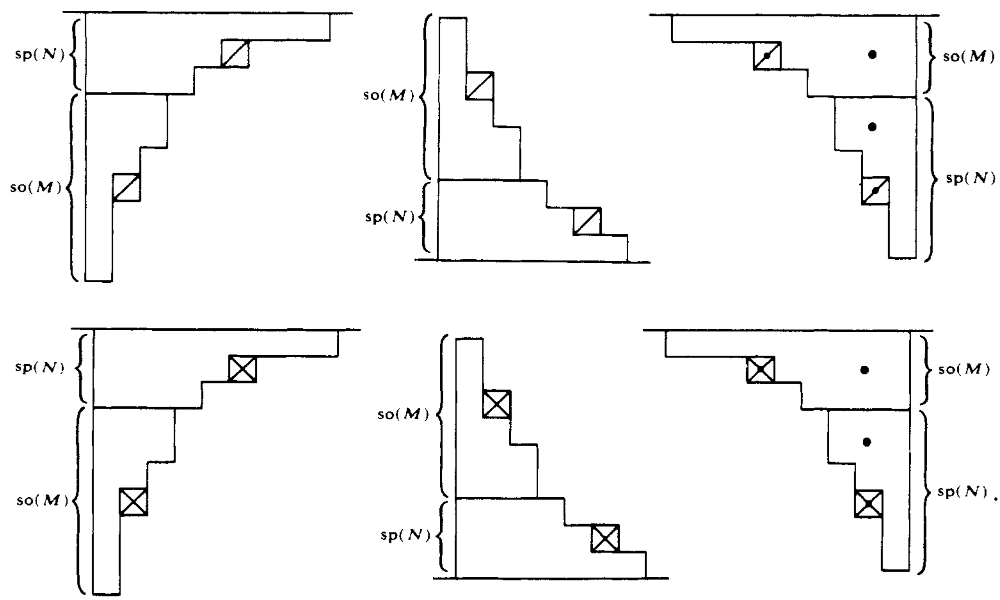
Remark 2. The same is true for the LW representations. For example



Remark 3. A su(m/n)-YST with $b_m = \bar{b}_n = 0$ (then $a_m = 0$) is obviously non-typical. We cannot however indicate it with a slash due to the lack of boxes in the last line. For this reason we cannot 'extract' the expected $I(\Lambda)$, implying the impossibility of this YST describing the MR. In general, such YST describe another SRR, or sometimes an IR, for example  for su(m/n), $m \neq n$. (When $m = n$ the YST of the adjoint is .) The same is true for the osp$(2/N)$ class of LSA, for example:



Remark 4. For the osp(M/N), $M \geq 3$, slashes and/or crosses may be written only when one box is in the sp(N) part and the other one in the so(M) part of the YST, giving for example:



6.2. *Equivalence of* YST

Two problems arise.

(*a*) The equivalence between two 'HW representations'-YST (or between two 'PHW'-YST).

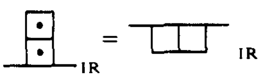
(*b*) The equivalence between an 'HW'-YT and 'PHW'-YST or more generally between two different kinds of 'PHW'-YST.

(i) su(m/n)

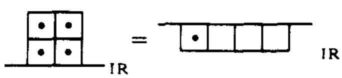
(*a*) Given a MR-YST, we already know that adding K columns of m undotted boxes and K columns of n dotted boxes leads to an equivalent YST.

If it describes a non-maximal representation we can also add N columns of dotted and undotted boxes, but if the starting YST do not possess crossed boxes, as in the second example of remark 1, we then have to put the crosses needed.

(*b*) We know that in principle only the IR can be described both as HW and LW representations. Even in that case, the following su(1/3) example:



but



shows that the relationship between upper contravariant and lower covariant indices and vice versa is not direct.

(ii) sp($2/N$).

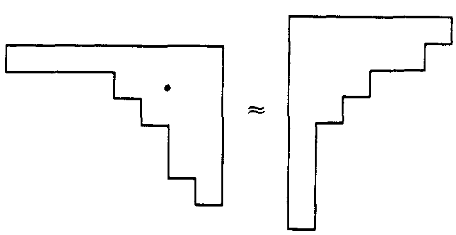
(*a*) For the MR, the columns of X undotted boxes, if $X \neq N$, are equivalent to columns of $N-X$ boxes, as for sp(N), but in contrast for the sp(N) case, if $X = N$, this is the combination of a column of one dotted box plus a column of N undotted boxes which is equivalent to a zero box.

(*b*) Same comment as for su(m/n).

(iii) osp(M/N)-YST, $M \neq 2$.

(*a*) In contrast to the type-1 LSA, two different 'HW'-YST (both made of undotted boxes) describe necessarily two inequivalent representations, since they cannot describe the same highest weight. The same is true for two 'PHW'-YST, both made of dotted boxes.

(*b*) For a given IR, there is necessarily equivalence between the 'HW'-YST and the 'PHW'-YST. When few boxes are involved, we have



For a more complete discussion of (*a*), see Deluc and Gourdin (1984).

7. YST, superdimension, index, etc

7.1. Superdimension of the su(m/n)-IR

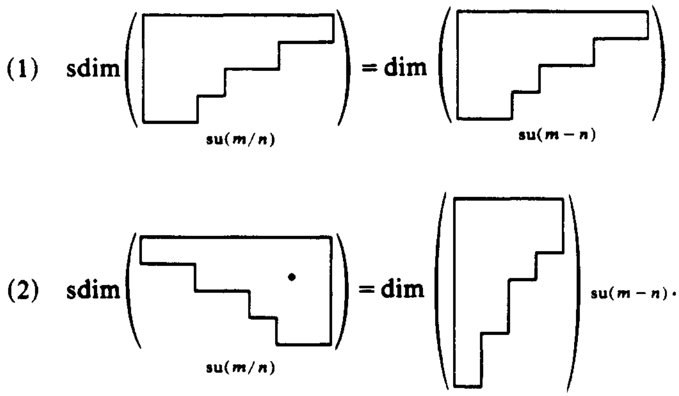
A way to compute the dimension of a su(n)-IR is the 'product of the boxes over the product of hooks' rule familiar to physicists.

The question is: is there a corresponding rule in the su(m/n) case?

Let $D \equiv \dim \square = m+n$ and $d \equiv \operatorname{sdim} \square = m-n$, then obviously $\dim \boxdot = D$ and $\operatorname{sdim} \boxdot = -d$, and it is easy to compute the following results if $m > n$:

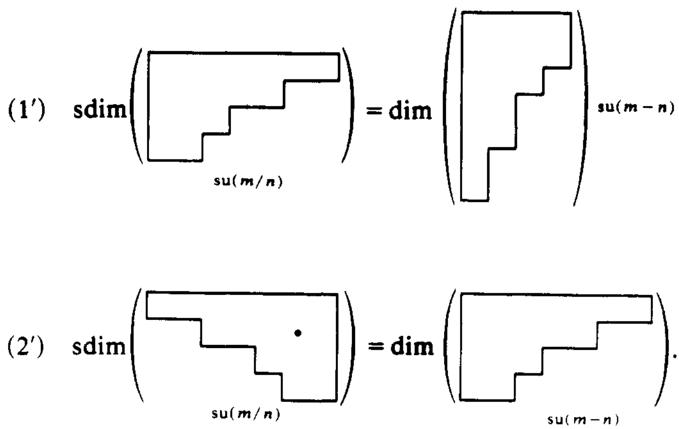
$$\begin{array}{lll}
\text{(i)} & \dim \square\square = \dim \begin{smallmatrix}\boxdot\\ \boxdot\end{smallmatrix} = \tfrac{1}{2}(D^2+d) & \operatorname{sdim} \square\square = \operatorname{sdim} \begin{smallmatrix}\boxdot\\ \boxdot\end{smallmatrix} = \tfrac{1}{2}d(d+1) \\
\text{(ii)} & \dim \begin{smallmatrix}\square\\ \square\end{smallmatrix} = \dim \boxdot\boxdot = \tfrac{1}{2}(D^2-d) & \operatorname{sdim} \begin{smallmatrix}\square\\ \square\end{smallmatrix} = \operatorname{sdim} \boxdot\boxdot = \tfrac{1}{2}d(d-1) \\
\text{(iii)} & \dim \boxdot\square = D^2-1 & \operatorname{sdim} \boxdot\square = d^2-1.
\end{array} \qquad (7.1)$$

Therefore, superficially, there is no 'hooks rule' for the dimension, but as we shall see, a very similar rule indeed exists for the superdimension. For example, we can verify on many examples that, if $m > n$, then



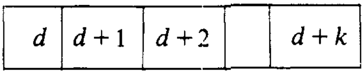
(7.2)

When $m < n$, rules (1) and (2) are now simply changed in

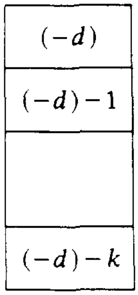


(7.3)

In fact, the flip in (2) or (1′) is artificial because of having $d<0$ or $-d<0$: the value of the 'product of the boxes' gives, for example, the equality



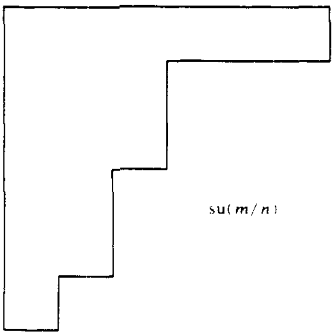
$= (-1)^{k+1}$



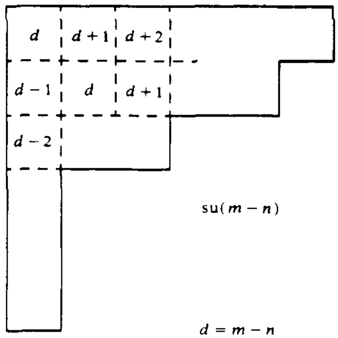
$= d \cdot d+1 \cdot d+2 \cdot \ldots \cdot d+k$

leading to an unified superdimension formula, $\forall \mathrm{su}(m/n)$ $m \neq n$:

(I) sdim

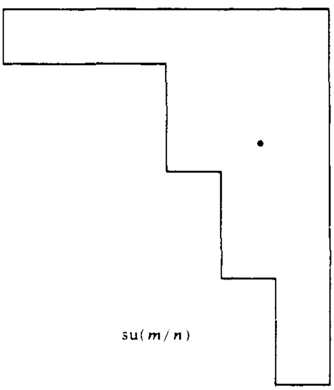


= dim

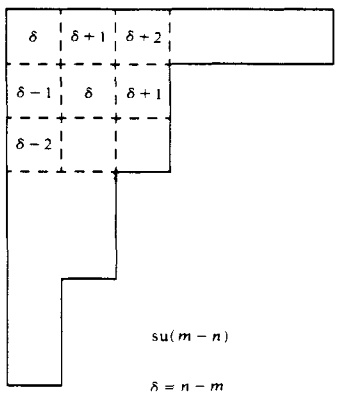


(7.4)

(II) sdim



= dim



(7.4′)

where d and δ are the numbers that have to figure in the product of the boxes.

The superdimension of the mixed YST can be computed with the aid of the third method: $V(\Lambda)_{\mathrm{IR}}$ has the same superdimension up to a sign as its maximal ideal $I(\Lambda)$ specified by the following HW, say ϕ, but this $I(\Lambda)$ may also be seen as the factor space $V(\phi)_{\mathrm{IR}}$ of $V(\phi)_{\mathrm{MR}}$, then sdim $V(\Lambda)_{\mathrm{IR}} = \mathrm{sdim}\ V(\phi)_{\mathrm{IR}}$.

This procedure can be repeated until we finally obtain sdim $V(\Lambda)_{\mathrm{IR}} = \mathrm{sdim}\ V(\Gamma)_{\mathrm{IR}}$ where the YST of $V(\Gamma)_{\mathrm{IR}}$ is made only of dotted or undotted boxes. Thus (I) or (II) can finally be used to compute the superdimension, giving a chain of equalities:

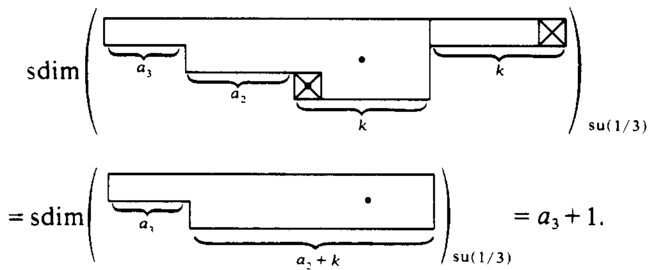
(III) $\mathrm{sdim}(\text{mixed YST})_{\mathrm{su}(m/n)} = \mathrm{sdim}(\text{mixed YST}')_{\mathrm{su}(m/n)}$

$$= \ldots = \mathrm{sdim}(\text{unmixed YST}'''')_{\mathrm{su}(m/n)}$$

$$= \dim(\text{YT}'''')_{\mathrm{su}(m-n)}.$$

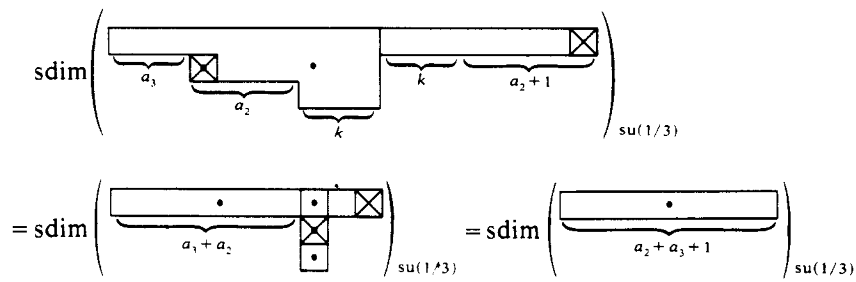
For example, we will consider again su(1/3) and its non-typicality conditions $a_1 = 0$, $a_1 = a_2 + 1$ and $a_1 = a_2 + a_3 + 2$.

(i) $a_1 = 0$:



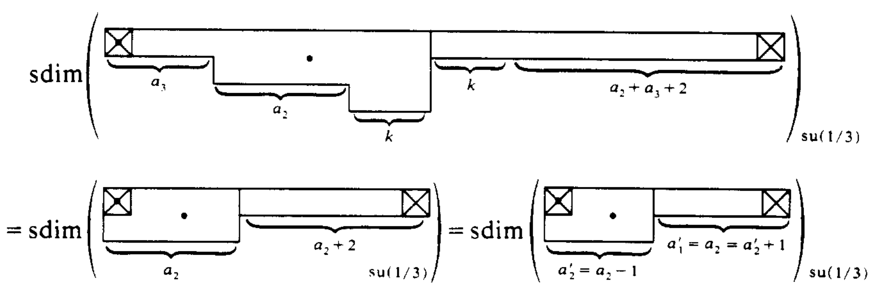
$= a_3 + 1.$

(ii) $a_1 = a_2 + 1$:



$= a_2 + a_3 + 2.$

(iii) $a_1 = a_2 + a_3 + 2$:



$= a_2' + a_3' + 2 \equiv a_2 + 1.$

These formulae for su(1/3) coincide with the ones given in Thierry-Mieg (1984).

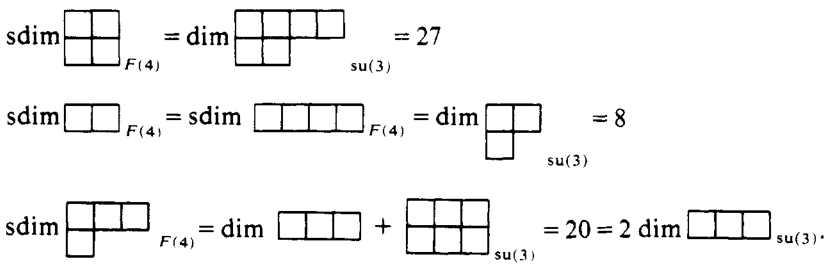
7.2. Superdimension for the other LSA

More generally, every superdimension formula for a given LSA can be expressed by a dimension formula for the related LA that we can find in the column 'sdim' in table A4 (for $su(n/n+1) \approx su(n+1/n)$ and $osp(2n+1/2n)$, the related LA is simply u(1)).

For example, consider the following $F(4)$ tensor product (see § 9):

$$40(8) \otimes 40(8) = 296(8) \oplus 756(20) \oplus 507(27) \oplus 40(8) \oplus 1.$$

The associated LA is su(3) where $8 \otimes 8 = 27 \oplus 10 \oplus 10^* \oplus 8 \oplus 8 \oplus 1$. Thus



Proof. For the infinite series of basic LSA, it was proven in Bars (1985) that the supercharacters $\chi(u)$ of the IR of these LSA are equal to the characters $\chi(u')$ of an IR of the related LA, but $\chi(1)$ gives the superdimension exactly. In particular, the formulation of (I) and (II) for $su(m/n)$ was proved in this way. Similarly, it was proved that the superdimension of an $osp(M/N)$-IR whose YST is made only of '$sp(N)$ boxes' is equal to the dimension of a $so(N-M)$-IR whose YT looks like the previous YST (see table A4 for the meaning of $so(N-M)$, when $N-M<0$).

Now, by using the third method, the superdimension of the other non-typical IR (whose YST have both boxes of the '$sp(N)$' and '$so(M)$' type) can also be found.

The proof for the exceptional LSA comes from Thierry-Mieg (1983a) where the following formula appeared (in an equivalent formulation):

$$\text{sdim } V(\Lambda)_G = k \cdot \dim V(\Lambda')_H \tag{7.5}$$

where $k=1$ when $G = su(1/n)$ or $osp(2/N)$, and where $k = 1$ or 2 when G is $osp(M/2)$, $D(2,1,\alpha)$, $G(3)$ or $F(4)$, and where the formula dim $V(\Lambda')_H$ is given by

$$\dim V(\Lambda')_H = \prod_{\alpha \in \Delta_s} (\Lambda + \rho_s, \alpha)/(\rho_s, \alpha)$$

where Δ_s represents the following set of roots: $\Delta_s \equiv \{\alpha \neq \beta | (\alpha, \beta) = 0$ where $\beta \in \Delta_1$ and $(\Lambda + \rho, \beta) = 0\}$, which is the root system of the related LA in that case.

Conjecture. The formula

$$\text{sdim } V(\Lambda)_G = k \cdot \text{sdim } V(\Lambda')_H \tag{7.5'}$$

which coincides with (7.5) for the above-mentioned LSA G is valid for any faithful $V(\Lambda)_{IR}$ of any basic LSA. The root system Δ_s of H, being specified exactly as before, is now in general the root system of a LSA.

By successive applications of the formula (7.5'), we should finally find

$$\text{sdim } V(\Lambda)_G = \underbrace{k' \cdot k'' \cdot \ldots \cdot k''''}_{k} \cdot \dim V(\Lambda'''')_{\text{related LA}}.$$

7.3. Index of a representation

The kinematic term of a LSA valued field

$$\Phi(x)=\sum_{i=1}^{\dim G}\Phi(x)^i T_i$$

where $T_i \equiv T(e_i)$ are the generators of a given representation $V(\Lambda)$, $e_i \in G$, has to be a symmetric bilinear invariant term. Thus it should be written $\frac{1}{4}\,\mathrm{STr}\,\partial_\mu\Phi(x)\partial^\mu\Phi(x)$, as the supertrace form is symmetric, bilinear and invariant. Two such forms are proportional for the basic LSA and this undetermined factor of proportionality is usually called the coupling constant (Schücker 1982).

However, contrary to the classical LA case where STr coincides with Tr, the trace form, the supertrace form may be degenerate. In that case, the index l_V of the representation V, defined by $\mathrm{STr}(T_iT_j)=l_V(e_i,e_j)$, is obviously vanishing and the superfield $\Phi(x)$ cannot propagate by virtue of the lack of a kinetic term and is only an auxiliary field of the theory.

A way to compute it, when sdim $G\neq 0$, is by using the following formula (Kac 1977):

$$l_{V(\Lambda)}=\mathrm{sdim}\;V(\Lambda)\cdot C_2(\Lambda)/\mathrm{sdim}\;G \tag{7.6}$$

where $C_2(\Lambda)$ is the eigenvalue $(\Lambda+2\rho,\Lambda)$ of the Casimir operator. (It can be computed with the help of the following formula: $(\Lambda+2\rho,\Lambda)=\Sigma_i\,\lambda_i(\Lambda+2\rho,\alpha_i)$. As for LA we have $\boldsymbol{\lambda}=A^{-1}\boldsymbol{a}$, where $\boldsymbol{a}$ is the highest-weight vector and A^{-1} is the inverse of the Cartan matrix. The A^{-1} are computed in Lemire and Patera (1982).)

The vanishing of the superdimension, and hence of $l_{V(\Lambda)}$, is easily recognised for the atypical $\mathrm{su}(m/n)$-IR with the help of (I), (II) and (III) and their analogue in the $\mathrm{osp}(M/N)$ cases.

In general, for type-1 LSA, if we know the G_0 content of $V(\Lambda)=V_k$, where, according to the Z gradation of these LSA, V_k is the subspace of $V(\Lambda)$ characterised by the u(1) eigenvalue $(q-k)$, we can quickly evaluate the vanishing of l_V, even when sdim $G=0$, by simply computing

$$\mathrm{STr}\;Q^2=\sum_k (q-k)^2(-1)^k\dim V_k \tag{7.7}$$

as $\mathrm{STr}\;Q^2=\dim G_1\neq 0$ for the adjoint representation.

In particular, (7.7) implies that we also have $\mathrm{STr}\;Q^2=0$ in all the typical cases, except when G is isomorphic to $\mathrm{su}(1/2)\approx\mathrm{osp}(2/2)$.

Proof.

$$\mathrm{STr}\;Q=\sum_k (q-k)(-1)^k\dim V_k=0$$

implies the following identity:

$$\sum_k (-1)^k k\dim V_k=q\cdot\mathrm{sdim}\;V(\Lambda)$$

which allows the alternative formulation of (7.7):

$$\mathrm{STr}\;Q^2=-q^2\,\mathrm{sdim}\;V(\Lambda)+\sum_k(-1)^k k^2\dim V_k.$$

In the typical case, $\dim V_k=\binom{N}{k}\dim V_0$, $N=\dim G_{-1}$, but $\Sigma_k^N(-1)^k k^2\binom{N}{k}=0$ when $N\geqslant 3$. Thus the second term vanishes as well as the first one, since sdim $V(\Lambda)_T=0$. Therefore $\mathrm{STr}\;Q^2=0$ in the typical case.

When $G = \mathrm{su}(1/2)$, then we find by simple calculation that $\mathrm{STr}\, Q^2 = 2 \cdot \dim V_0(\Lambda)$ in the typical case. Therefore any faithful IR of this LSA admits a non-degenerate supertrace form since in the non-typical case we have $\mathrm{STr}\, Q^2 = \pm n(n \pm 1)$ when $\dim V(\Lambda) = 2n \pm 1$. Finally, note that we cannot evaluate l_V in this way in the case of $\mathrm{su}(n/n)$ since $\mathrm{STr}\, C^2 = c \cdot \mathrm{STr}\, V(\Lambda) = 0$, $\forall V(\Lambda)$.

7.4. *Self-conjugated representations*

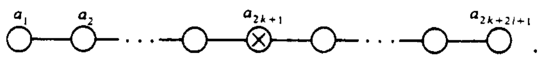
A representation $V(\Lambda)$ is self-conjugate if for every weight σ belonging to $V(\Lambda)$, the weight $-\sigma$ also belongs to that $V(\Lambda)$. In that case, a representation of a real Lie algebra is either real or quaternionic.

In particular, a $\mathrm{su}(n)$ IR is self-conjugate if the Dynkin indices satisfy $a_i = a_{n-i}$ $\forall i$; and similarly a representation of $\mathrm{so}(4m+2)$-IR is self-conjugate iff $a_{n-1} = n$. More simply, all the IR of the other Lie algebra are self-conjugate.

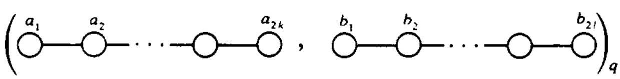
Consequently, for the type-2 LSA, except $\mathrm{osp}(4m+2/2n)$, any representation $V(\Lambda)$ is self-conjugate. For the $\mathrm{osp}(4m+2/2n) \approx D(2m+1, n)$, a representation is self-conjugate iff $a_{2m+n-1} = a_{2m+n}$. Similarly in the $C(2m+1, n)$ system of simple roots, we have to compare a_{2m+n+1} with the Dynkin index d.

For the type-1 LSA, a typical representation is self-conjugate if the representation of the ground floor is conjugate to the one of the Nth floor ($N = \frac{1}{2} \dim G_{\bar{1}}$). It is not hard to see that none of the typical representations of $\mathrm{su}(2k+1/2l+1)$ and $C(n)$ can be self-conjugate, since the necessary condition $q = N/2$ implies a non-typicality condition.

Proof for $\mathrm{su}(m/n) = \mathrm{su}(2k+1/2l+1)$:

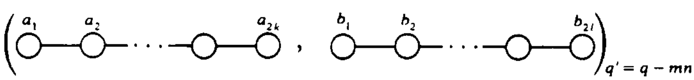


Let $V(\Lambda) \equiv \bigoplus_{i=0}^{mn} V_i$ be a typical IR where the V_i are the subspaces of $V(\Lambda)$ characterised by the Q eigenvalue $q - i$; thus

$V_0 =$


$b_j = a_{2k+1+j}$

and

$V_{mn} =$


where

$$q = \frac{1}{n-m}\left(n \sum_{i=1}^{2k} i \cdot \alpha_i + mn \cdot a_{2k+1} - m \sum_{j=1}^{2l} j \cdot b_{2l+1-j}\right). \tag{7.8}$$

Now $V(\Lambda)$ is self-conjugate iff

$$a_i = a_{2k+1-i} \tag{7.9}$$

$$b_j = b_{2l+1-j} \tag{7.10}$$

$$q' = -q \qquad \text{i.e. } q = \tfrac{1}{2}mn. \tag{7.11}$$

By introducing (7.9) and (7.10) in (7.8) we get

$$q = \frac{mn}{n-m}\left(\sum_{i=k+1}^{2k} a_i + a_{2k+1} - \sum_{j=1}^{l} b_j\right) \tag{7.12}$$

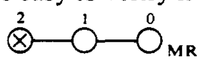
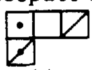
which, together with (7.11), gives

$$\sum_{i=k+1}^{2k} a_i + 2_{2k+1} - \sum_{j=1}^{l} b_j = \tfrac{1}{2}(n-m) - (l-k)$$

but in contradiction to this hypothesis, this is precisely the non-typicality condition

$$\sum_{i=k+1}^{2k} (a_i + 1) + a_{2k+1} - \sum_{j=1}^{l} (2_{k+1+j} + 1) = 0.$$

The proof for osp(2/2n) is similar.

In the non-typical case, by using the third method, it is easy to verify if an invariant subspace can be self-conjugate. As an example, consider  $_{\mathrm{MR}}$, whose YST is . Thus at the ground floor we have $V_0(\Lambda) = 3_{+2}$ of su(3)+u(1), whence at the third floor we have a 3_{-1}. The YST of this $V(\Lambda)_{\mathrm{MR}}$ clearly shows that a $I(\Lambda)$ is self-conjugate since the ground floor of $I(\Lambda)$ is in a 3^*_{+1} of su(3)+u(1).

8. Tensor products

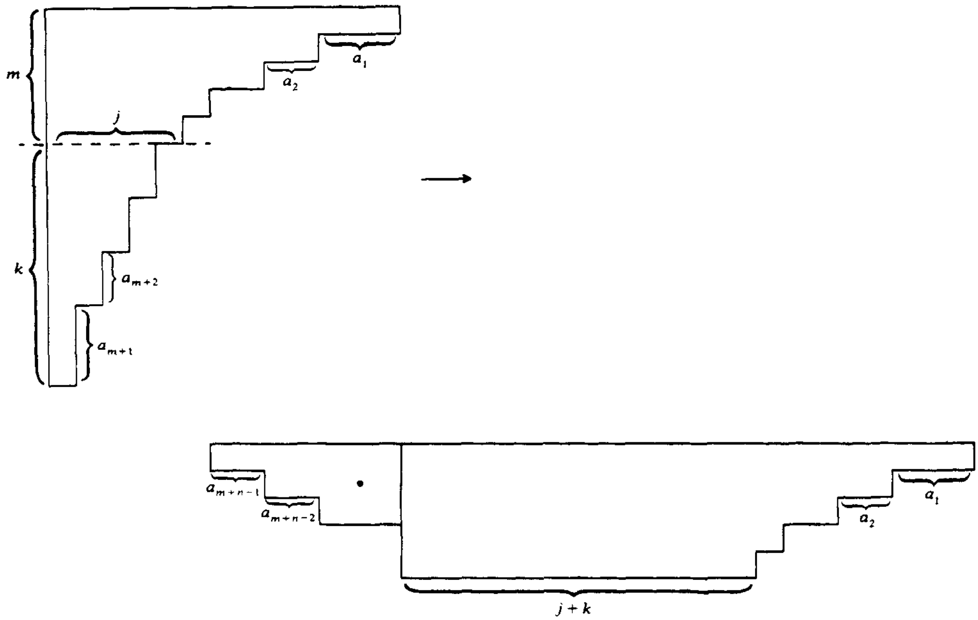
This section is just an overview of the difficulties (Rittenberg *et al* 1977) that appear for finding the tensor product rules in the su(m/n) case. The difficulties are certainly more important for the other LSA.

The YST that appear when the reductions of the tensor products are made will then be called 'tensor'-YST (T-YST), as opposed to the '(P)HW'-YST already defined in this paper. If we identify the T-YST with the HW-YST for the fundamental IR □ and similarly with the conjugate fundamental one ⊡, then any su(m/n) HW-YST is at the same time a consistent T-YST. The converse is not always true, but in that case the link between the two notations is in fact very simple, as we shall see.

8.1. *Tensor product of su(m/n) non-typical IR*

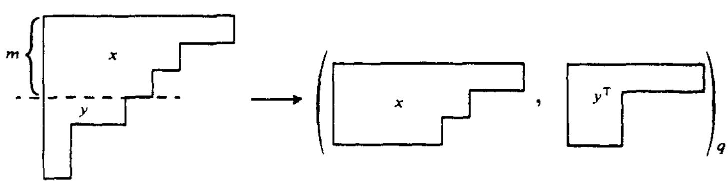
When making the reduction of the tensor product of two non-typical YST both made of undotted boxes, we simply have to follow the same rules as the ones for su(N), $N >$ total number of boxes considered (Bars *et al* 1983, Dondi and Jarvis 1981, King

1982). The resulting T-YST that have columns with more than m boxes can be reinterpreted as HW-YST, according to



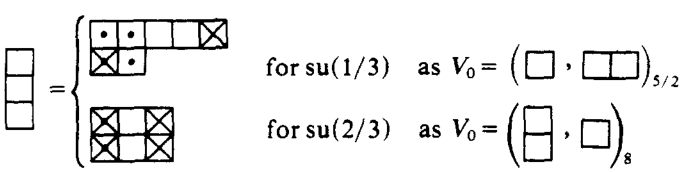
(8.1)

This observation is obtained from the supersymmetrisation rule (Bars *et al* 1983) that allows us to find $V_0(\Lambda)$. Hence the HW-YST:

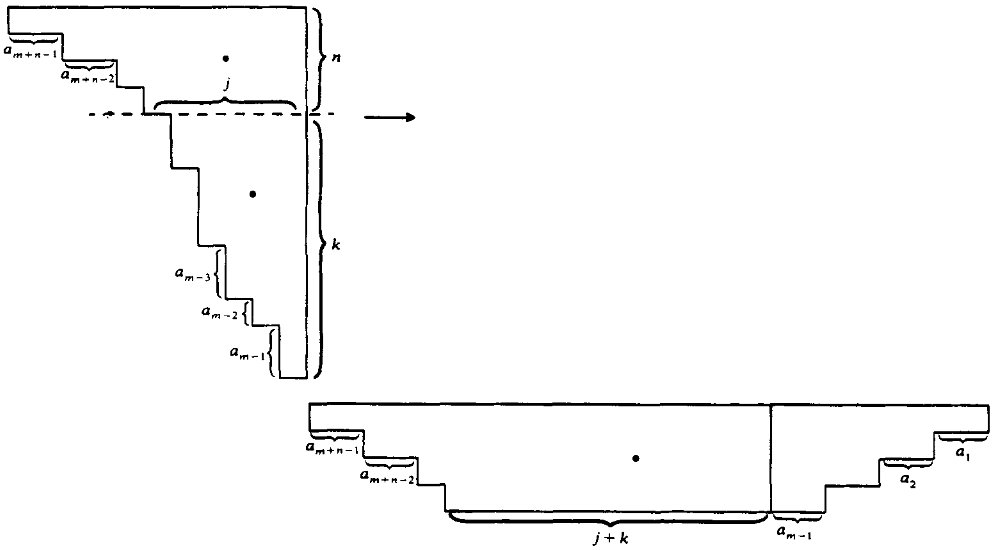


with $q = [(n-m)^{-1}(n \cdot j)] - k$, where j is the total number of boxes and k is the number of boxes under the mth line.

The crosses and slashes must not be forgotten, as in the following example crosses must be added:

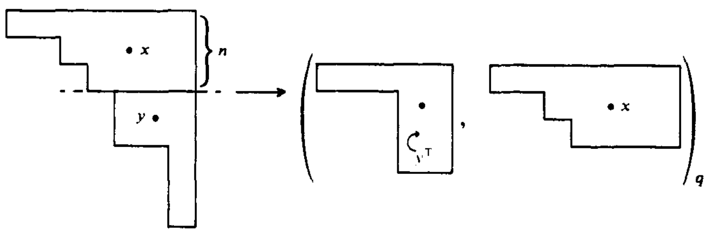


Similarly, when making the reduction of the tensor product of two non-typical YST both made only of dotted boxes, the YST that have columns with more than n boxes have to be changed according to the following rule:



(8.2)

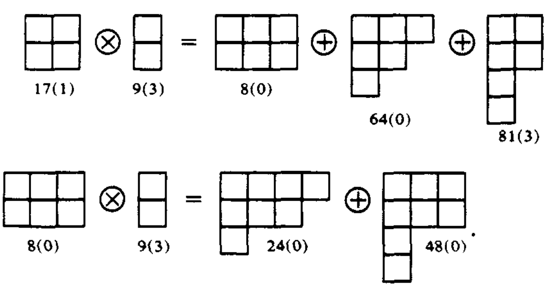
$V_0(\Lambda)$ being specified this time according to



with $q=[(n-m)^{-1}(-m\cdot j)]-k$ where j is the total number of dotted boxes and k is the number of boxes under the nth row.

When an arbitrary mixed T-YST is considered, the corresponding HW-YST is found by gluing together the identical columns obtained by applying (8.1) for the undotted part of the T-YST and (8.2) for the dotted part, respectively.

A T-YST is illegal if the corresponding HW-YST is ill-defined. It should simply be eliminated when it appears in the reduction of the tensor products. For example, consider the following two su(1/3) tensor products:

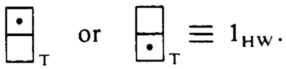


Thus, by a simple dimensional argument, the T-YST  cannot describe any representation, which is fortunate since it is illegal.

Remark. The only differences with Bars *et al* (1983) are that the dotted part of the YST has to be transposed, and crosses and slashes should be added in order to have a well defined non-typical HW-YST. Note that if the first difference may reflect different attitudes in regard to the apparent symmetry–antisymmetry flip discussed in the former section; the / in  or  boxes of the above-mentioned reference have nothing to do with the invariant subspaces; they simply denote the 'super' indices.

The difficulties really appear when the tensor products are made with both dotted and undotted boxes. By simply making the tensor product  we see that dotted and undotted boxes are on an equal footing and we have to interpret the T-YST according to

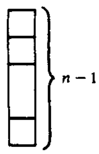
and (8.3)



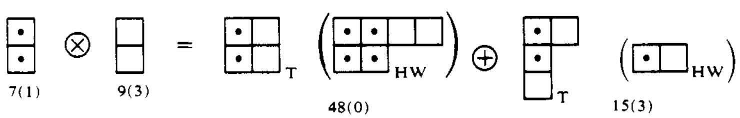
When the tensor product of an arbitrary YST is made with a second one made of only one box, it seems, as is shown in appendix 3 for su(1/3), that the reduction follows the following simple rules:

(i) add the box of the second YST to the first one in all possible ways,

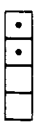
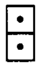
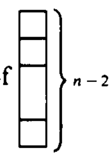
(ii) simplify the resulting YST with the help of (8.3),

(iii) keep one copy among all the equivalent YST, two YST being equivalent by permutation of a column of any number of dotted boxes with another one made of undotted boxes. From that point of view,  plays a role similar to  $n-1$ in su(n).

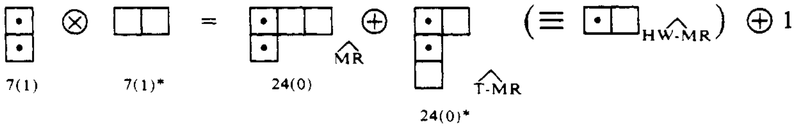
However, this quite simple rule is already hard to generalise when the second YST is made of two boxes. For example, for su(1/3) we have



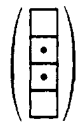
(8.4)

but the expected  do not appear. Therefore  is not the full analogue of  $n-2$ and we need a point (iv) explaining that fact.

Unfortunately these four points are far from being sufficient. For example,

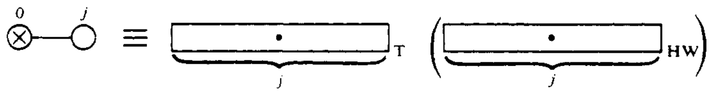


(8.5)

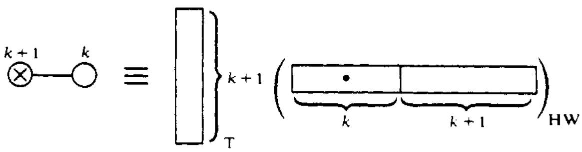
where $\widehat{\text{MR}}$ denotes a representation having the same weights as the maximal SRR one, as we can check by weight techniques or dimensional arguments. However the algorithm does not specify if we have a direct or semidirect sum of non-typical IR. Thus we need a point (v) that will save us from computing explicitly the Clebsch–Gordan coefficient, and a point (vi) for explaining what is the 1  .

8.2. Another partial algorithm

The tensor product of the non-typical IR of su(1/2):

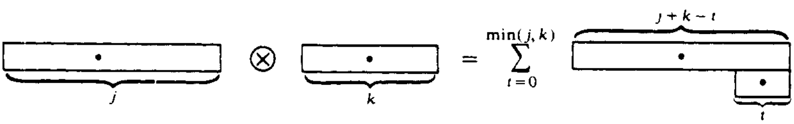


and



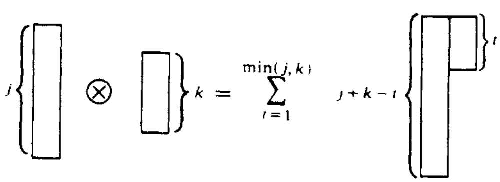
is unambiguously found using weight techniques. Indeed, in any case, only one non-typical IR appears (with many typical IR). Thus only the direct summand of IR is possible when the reduction is made.

According to the algorithm of § 8.1, we find



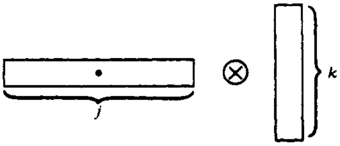
(8.6)

and



(8.7)

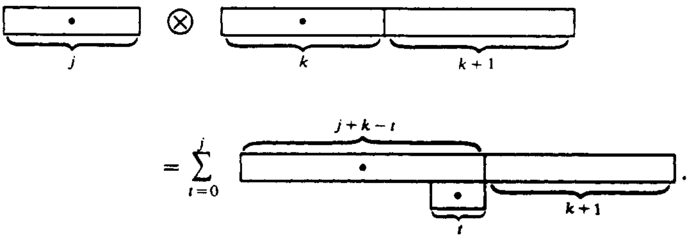
in agreement with the weight techniques, as is proved by the matching of the dimensions. But, as we have seen, the rules for the making of



are not well defined.

In contrast, by now using the HW-YST, the correct answer is easily written when $j \leqslant k$.

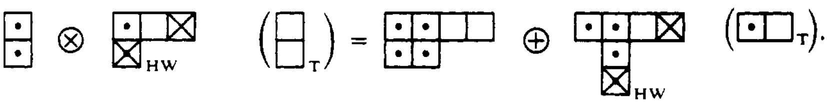
Let  then, if $j \leqslant k$,



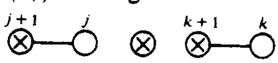
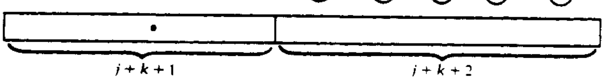
(8.8)

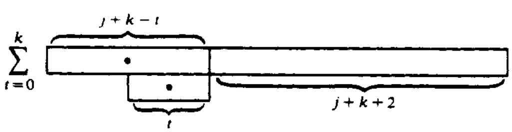
Therefore, apparently, when the tensor product with HW-YST is made we have (a) to combine undotted boxes as for su(m) and dotted boxes as for su(n), but (b) we do not have to combine dotted boxes of the first YST with the undotted boxes of the second YST and vice versa. Furthermore (c) in contrast to the T-YST algorithm, we have to discard the YST that appear with a column having more than m undotted boxes or more than n dotted boxes.

According to this HW-YST algorithm, we get the correct result for (8.4):



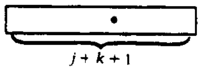
But unfortunately, due to point (c), this algorithm is still incomplete for (8.5)

This algorithm also fails for , $j \geqslant k$, since we have to add now  to the result obtained according to the HW-YST algorithm:

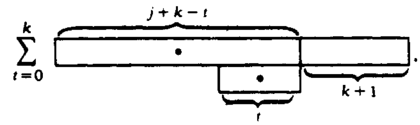


(8.6′)

and similarly for (8.8) when $j > k$ since the correct result necessitates adding



to



(8.8′)

Therefore the only reasonable conclusion is to say that the problem of the reduction of the tensor products is far from being resolved, especially because of the absence of a method able to determine quickly for which type of summand the non-typical irreducible subspaces appear.

In Marcu (1980b), the reduction of some tensor products of su(2/1) ($\approx$su(1/2)) representations have been extensively studied. The reader can easily translate the non-degenerate tensor product into the YST language, knowing that the convention for the u(1) eigenvalue is half of that used in the present paper and that we have the following correspondences.

(i) IR

$$[q]_+ = \overset{2q-1}{\bigcirc}\!-\!\overset{-2q}{\otimes} \qquad [q]_- = \overset{2q}{\bigcirc}\!-\!\overset{0}{\otimes} \qquad q = \tfrac{1}{2}Z_+$$

$$[b, q],\ b \neq \pm q = \overset{2q-1}{\bigcirc}\!-\!\overset{2(b-q)}{\otimes}$$

(ii) HW-MR

$$[q-\tfrac{1}{2}, q]_+ = \overset{2q-1}{\bigcirc}\!-\!\overset{-2q}{\otimes} \qquad [q, q-\tfrac{1}{2}]_- = \overset{2q-1}{\bigcirc}\!-\!\overset{0}{\otimes}$$

(iii) LW-MR

$$[q, q-\tfrac{1}{2}]_+ = ([q-\tfrac{1}{2}, q]_+)^* \qquad [q-\tfrac{1}{2}, q]_- = ([q, q-\tfrac{1}{2}]_-)^*.$$

This paper also provides important examples of SRR that appear in degenerate tensor products and which are not (P)HW-SRR. These SRR, made of three or four irreducible subspaces, are explicitly realised in a related paper (Marcu 1980a), for example

$$[q-\tfrac{1}{2}, q+\tfrac{1}{2}, q]_\pm = [q-\tfrac{1}{2}]_\pm \supset\!\!\!\!+ [q]_\pm \subset\!\!\!\!+ [q+\tfrac{1}{2}]_\pm$$

$$[q, q-\tfrac{1}{2}, q+\tfrac{1}{2}]_\pm = [q-\tfrac{1}{2}]_\pm \subset\!\!\!\!+ [q]_\pm \supset\!\!\!\!+ [q+\tfrac{1}{2}]_\pm$$

$$[q, q\pm 1, q\pm\tfrac{1}{2}, q\pm\tfrac{1}{2}]_\pm = [q]_+ \supset\!\!\!\!+ [q\pm\tfrac{1}{2}]_+ \subset\!\!\!\!+ [q\pm 1]_+ \supset\!\!\!\!+ [q+\tfrac{1}{2}]_+$$

$$[q, q-\tfrac{1}{2}, q+\tfrac{1}{2}, q]_\pm = [q]_- \begin{array}{ccc} \otimes & [q-\tfrac{1}{2}]_+ & \otimes \\ & & \\ \otimes & [q+\tfrac{1}{2}]_+ & \otimes \end{array} [q]_-$$

etc.

Since any su(2/1) (P)HW-SRR is made, at the most, of the direct summand of two IR, we believe these strange SRR to be examples of representations built by induction on a finite-dimensional indecomposable representation of the non-semisimple sub-superalgebra $P = G_0 + G_{+1}$ of $G \approx$ su(2/1). In the above-mentioned reference, Marcu also obtained representations with non-diagonal Cartan subalgebra, probably of the type $\mathrm{Ind}_{u(1)}^{su(2)}\,\varphi$ where φ is a non-trivial u(1) module.

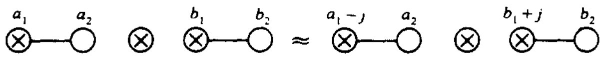
In the general case, many SRR which are not (P)HW-SRR can be realised in these ways, as any LSA can be chosen in place of $G \approx su(2/1)$.

8.3. *Tensor product with a typical* IR

As soon as we have consistent rules for the tensor product of irreducible non-typical IR, it is quite easy to find the weights that appear when tensor products are made of typical representations. Indeed, when the value of the Dynkin index a_m is changed, we simply shift uniformly the u(1) eigenvalues. In particular, when the new a_m coincide with a non-typical value, the shifted weights belong to (direct and/or semidirect) sums of non-typical IR. We can then apply the above, presumably consistent, rules for the tensor product. The inverse shift on all the weights that have appeared remains to be made as well as their final regrouping into IR.

It does not solve, however, the problem of knowing which kind of summand we have to take for the non-typical IR.

The presence of semidirect summands when some tensor products of typical IR are made may look strange, but can be easily understood. Indeed, since the following isomorphism holds:

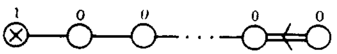


$(\approx V)$ (8.9)

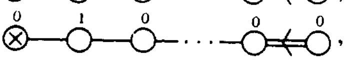
when there are only typical IR in V, it is natural to suppose that (8.9) still holds when non-typical representations also appear in V. But by choosing some specific value for j, this V can arise either from the tensor product of non-typical SRR (and in that case, it is easily conceivable that some SRR are present in V) or simply from the tensor product of typical IR.

8.4. T-YST *for the other basic* LSA

As for $su(m/n)$, the (P)HW-YST defined in the previous subsections for the type-2 LSA apply equally well as T-YST. In contrast, the HW-YST of the osp$(2/2n)$ do not give well defined YST. For example, $\square_{HW}$ does not describe the fundamental IR

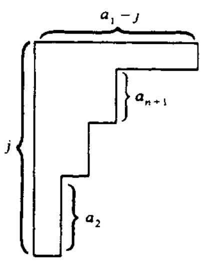


, but in contrast the non-typical IR

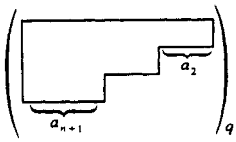


, and $\boxdot_{HW}$ describes a 2^{2n}-dimensional typical IR.

The T-YST of $C(n+1) \approx$ osp$(2/2n) \approx C(1, n)$ can be defined by analogy with those of the other osp$(2m/2n) \approx C(m, n)$. When the value of a_1 is sufficiently high, we can define

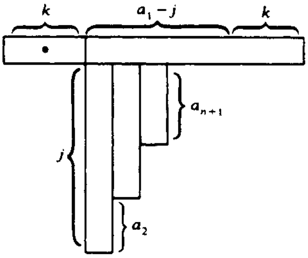


(8.10)

if we admit that this YST implies $V_0(\Lambda) =$  where $q = a_1 - \Sigma_{i=2}^{n+1} a_i$

is by definition the total number of boxes minus the number of boxes under the first row.

When a_1 is too small, a possible definition is



(8.11)

$V_0(\Lambda)$ being defined as before with q the total number of undotted boxes, minus the number of dotted boxes, minus the number of boxes under the first row.

Now $\square_T$ describes the fundamental IR, but $\boxdot_T$ is still used to describe the above 2^{2n}-dimensional IR. Since we have that dim $\square_T \neq$ dim $\boxdot_T$ and that $\square_T$ is self-conjugate, the suppression of the dots in (8.14), as well as in Morel *et al* (1985) is probably justified.

On these T-YST, the non-typicality condition $a_1 - \Sigma_{i=2}^{j} (a_i + 1) = 0$ is equivalent to

$$b_1 - \bar{b}_1 = c_j + (j-1) \tag{8.12}$$

and should be noted as for the type-2 LSA in order to find the invariant subspaces.

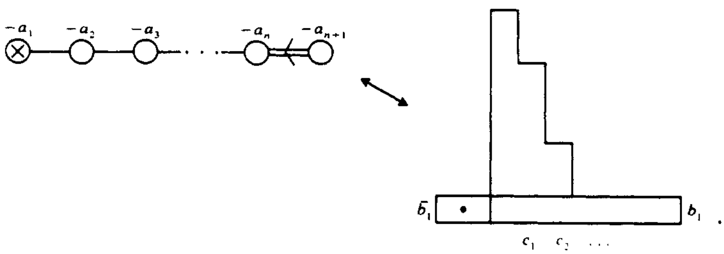
Contrary to the HW-YST case, the other non-typicality conditions, such as some of the ones of $F(4)$ and $G(3)$ as we shall see, are hard to write for the T-YST. However, for osp$(2/2n)$, we can still find the invariant subspaces relatively easily without using the HW-YST knowing that

(i) $V(\Delta, \Lambda)_{\mathrm{IR}} = I(-\Delta, \mu)$ where $\mu = 2(q/2n - 1)\rho_1 - \Lambda^*$,

(ii) if Λ implies that

$$a_1 - \sum_{i=2}^{j} (a_i + 1) - 2 \sum_{i=j+1}^{n} (a_i + 1) = 0$$

then the Dynkin indices a_i' of $I(-\Delta, \mu)$ satisfy $a_1' - \Sigma_{i=2}^{j} (a_i' + 1) = 0$. Thus we can use (8.14), adapted for the LW-T-YST:

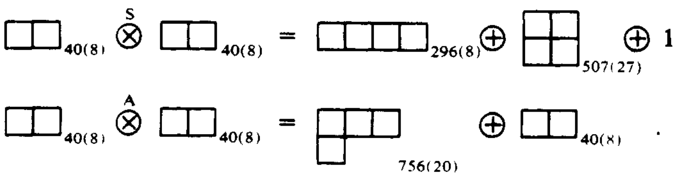


(8.13)

9. YST for the exceptional LSA

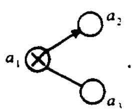
In contrast to the infinite series of basic LSA, $D(2, 1, 2)$ and $D(2, 1, 3)$ excepted, the adjoint is always the smallest representation of the exceptional LSA. Consequently, there is no convenient way to have YST with only one box: for example, if we had denoted by $\square$ the adjoint 40(8) of $F(4)$ (the number in brackets denotes the superdimension) then $\square\square \equiv (\square \overset{S}{\otimes} \square)$ should have denoted the completely reducible representation $296(8) \oplus 507(27) \oplus 1$, and similarly $\begin{smallmatrix}\square\\\square\end{smallmatrix} \equiv (\square \overset{A}{\otimes} \square)$ should have denoted the completely reducible $756(20) \oplus 40(8)$. But it is hard to see the 507 and the 756 resulting from some kind of contractions on the indices. In fact, this last remark indicates clearly that we should seek true T-YST which could be at the same time well-behaved HW-YST, like the ones of the osp(M/N), $M \neq 2$.

The above-mentioned problem disappears if we assign to the 40(8) the following YST: $\square\square$. Then



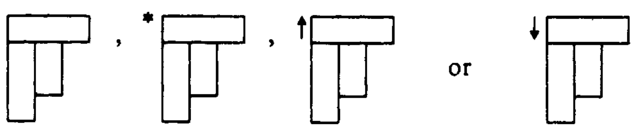
The price of this is that the consistency conditions cannot be 'understood', as for the type-2 osp(M/N), by simply seeing the YST. If, as we shall see, the notations for some of the non-typicality conditions remain quite similar to the osp(M/N), $M \neq 2$, case, the other non-typicality conditions do not have clear interpretations. This strongly suggests that the following YST of the exceptional LSA are the analogues of the $C(n)$ T-YST, but not of the (P)HW-YST.

(*a*) $D(2, 1, \alpha)$



Consulting table A12, it is clear that non-typical representations exist only for rational values of α, which may be chosen to be larger or equal to 1, according to table A2.

In fact, due to the infinite possible values of α, we will not discuss the invariant subspaces which depend crucially on this parameter α. For example, the dimension of the smallest representation is 6, 10, 14 for $\alpha = 1, 2, 3$, respectively, and 17 for the other values of α (Thierry-Mieg 1983b). We will simply admit that a YST has one of the following shapes:



when $V_0(\Lambda)$ is  of su(2)+so(4), and where ◆ eventually denotes *, ↑ or ↓.

The proof that $D(2,1,1)$ alone has a fundamental representation (six dimensional) is quite simple: the Cartan matrix of $D(2,1,\alpha)$ is

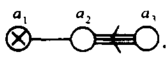
$$\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Thus $[E_{\alpha_2}, E_{\alpha_3}]=0$. When $b=1$, then we must have for consistency that

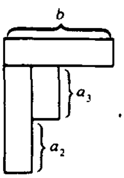
$$E_{-(\alpha_1+\alpha_2)}E_{-\alpha_1}|\Lambda\rangle \equiv 0 \equiv E_{-(\alpha_1+\alpha_3)}E_{-\alpha_1}|\Lambda\rangle.$$

Thus we should satisfy simultaneously $a_1\cdot[a_1-(a_2+1)]=0$ and $a_1\cdot[a_1-\alpha(a_3+1)]=0$; hence $a_1=a_2+1=\alpha(a_3+1)$, in agreement with table A9. But $V(\Lambda)$ can be the fundamental IR only if $a_2=a_3=0$; hence we need $\alpha=1$. But $D(2,1,1)\approx D(2,1)$ is not an exceptional LSA and is not the subject of this section.

(*b*) $G(3)$

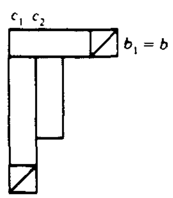


We simply propose

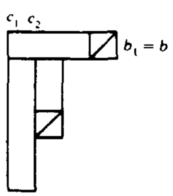


Clearly, the consistency conditions corresponding to $b=1$ (impossible) and $b=2$ ($a_2=0$) obviously have no 'YST' interpretation. In contrast, the following non-typical YST are denoted in a traditional way:

when $2b=c_1+2c_2+6$

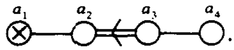


when $2b=2c_1+c_2+7$



but the non-typicality condition $2b=c_1-c_2+6$ corresponding to $(\Lambda+\rho, \alpha_1+3\alpha_2+\alpha_3)=0$ cannot be specified in such a way. The reason is that $\chi=\Lambda-\alpha_1-3\alpha_2-\alpha_3$ cannot be a highest weight of an invariant subspace with respect to $su(2)+G_2$. Thus *a fortiori* it cannot be the HW of a $G(3)$-invariant subspace.

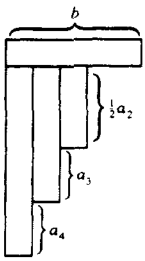
(*c*) $F(4)$



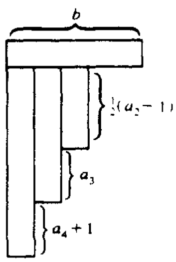
The most difficult YST to find were those of $F(4)$. Firstly, by looking at the $su(2)\oplus so(7)$ content of some $V(\Lambda)_{IR}$, according to the Thierry-Mieg *table of representations* (see Thierry-Mieg 1983a), it was possible to suspect the existence of spinorial IR for $F(4)$, the spinor or tensor type not being related to the so(7)-IR, part of the definition of $V_0(\Lambda)$, but in contrast to the so(7)-IR associated with the singlet of the su(2). Supporting this idea, the spinorial representations identified in that way are always typical, like the $osp(M/N)$.

Finally, it was possible to define quite well behaved YST to finding the irreducible subspaces of the tensor-MR. The spinor-YST are completed with an arrow, as for $so(N)$.

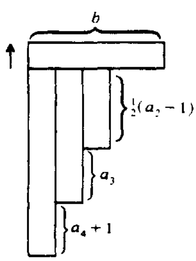
(i) Tensor-YST, when $b+a_2$ is even (i.e. when a_1 is an integer, since $b=2a_1-3a_2-4a_3-2a_4$). More precisely we find, when both b and a_2 are even,



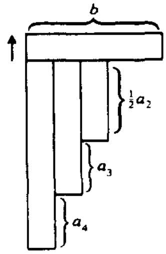
or when both b and a_2 are odd



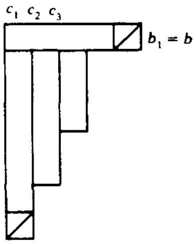
(ii) Spinor-YST, when $b+a_2$ is odd (or a_1 a half-integer), b even, a_2 odd:



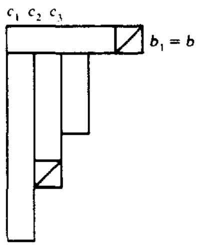
b odd, a_2 even:



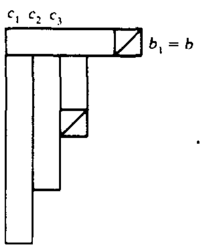
The non-typicality conditions can be written in the following way: when b is even *and* $a_1-2a_2-4a_3-3a_4-6=0$, i.e. $3(b-4)=2(c_1-c_2+c_3+1)=0$, then



when b is even *and* $a_1-2a_2-4a_3-2a_4-8=0$, i.e. $3(b-4)=2(c_1+c_2-c_3+1)=0$, then



and when b is odd *and* $a_1-3a_2-4a_3-2a_4-9=0$, i.e. $3(b-4)=2(c_1+c_2+c_3)+1=0$, then



As for $G(3)$ the other non-typicality conditions cannot be written in such a way. Thus, in contrast to the osp(M/N), $M\neq 2$, case (and the su(m/n) case), it is impossible for the exceptional LSA to find T-YST which are, at the same time, completely well behaved as (P)HW-YST. The situation is in fact more similar to the osp$(2/N)$ case, except that now a true definition of (P)HW-YST is still missing, mainly due to a lack of explicit knowledge about the 'large' representations.

10. Conclusions

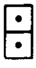
(i) In this paper we have recalled some well known results about the IR and the SRR and we have given many concrete examples and explicit general statements in a formulation accessible to the non-specialist; in particular, we have shown how to characterise algebraically the possible invariant subspaces of the non-typical representations and we have reinterpreted the so-called consistency conditions as necessary non-typicality conditions for obtaining finite-dimensional representations.

(ii) The algebraic characterisation of atypicality which leads, through the third method, to the intuitive 'contraction of the indices' interpretation can be analysed more precisely in terms of the basis vectors of the representation when tensor products are made. Indeed, some of them become unexpectedly collinear (Marcu 1980b) for a given set of Dynkin indices (or the Casimir operator eigenvalues (Rittenberg *et al* 1977, Scheunert 1984)), giving in that way a geometric interpretation of the non-typicality.

(iii) We have presented for the first time, to our knowledge, YST for the exceptional LSA. In addition, the formulation of all our su(m/n) and osp(M/N)-YST carry more information than the previously defined ones, the possibility of defining YST which are able to characterise inequivalent representations having the same HW or PHW being throughly verified. In particular, the (P)HW-YST permit us to find graphically the invariant subspaces of the SRR.

(iv) These (P)HW-YST permit a relatively quick computation of the superdimension of the non-typical IR and can be easily related to T-YST, which have superficially good tensorial properties. In fact, the T-YST are almost identical to the YST previously defined in other papers. Note that, for $C(n)$, the (P)HW-YST are the only ones that carry enough information to be able to characterise the different representations specified by the same highest weight. This will probably also be the case for the exceptional LSA, but (P)HW-YST remain to be exhibited for these LSA.

(v) We have shown that, in the non-typical case, different choices of the simple roots, i.e. different choices of positive/negative roots implying corresponding 'generalised' notions of highest weight, lead to inequivalent finite SRR having the same weights. Furthermore, we have shown that for the type-1 LSA a given SRR and its conjugate cannot both be defined as HW-SRR with respect to the same system of (pseudo-)positive roots.

(vi) The choice of positive/negative roots can be even more decisive in the $P(n) \equiv P(n)_{-1} + P(n)_0 + P(n)_{+1}$ class of LSA, where $P(n)_{-1}$, $P(n)_0$ and $P(n)_{+1}$ are, respectively, ,  and  of su$(n+1)$. As a $P(n)$ is a type-1 LSA (its odd part is completely reducible), we expect that any of its HW-MR, say $V(\Lambda)^+_{\mathrm{MR}}$, is made as

$$V_0(\Lambda)\otimes\left(\sum_{k=0}^{N}\Lambda^k P(n)_{-1}\right) \qquad N=\dim P(n)_{-1}=\tfrac{1}{2}n(n-1)$$

thus implying $\dim V(\Lambda)^+_{\mathrm{MR}} = 2^N \dim V_0(\Lambda)$. In contrast, starting from the same $V_0(\Lambda)$, a LW-MR is expected to be

$$V_0(\Lambda)\otimes\left(\sum_{k=0}^{N'}\Lambda^k P(n)+1\right) \qquad N'=\dim P(n)_{+1}=\tfrac{1}{2}n(n+1)$$

thus

$$\dim V(\Lambda)^-_{\mathrm{MR}} = 2^{N'}\dim V_0(\Lambda) = 2^n \dim V(\Lambda)^+_{\mathrm{MR}}.$$

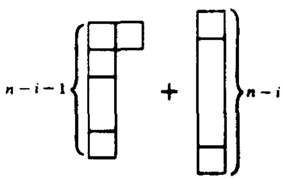
(vii) It is interesting to see the great similarity between the Z grading of the $\mathrm{su}(n/1)$ representation $\overset{1}{\bigcirc}-\overset{0}{\bigcirc}-\cdots-\overset{0}{\bigcirc}-\overset{-n}{\otimes}$ and the Z grading of the Cartan-type LSA $W(n)$ and two of its subsuperalgebras: $S(n)$ and $\tilde{H}(n)$.

We simply recall that $W(n) \approx \mathrm{der}\, \Lambda(n)$ where $\Lambda(n)$ is the Grassmann superalgebra with n basic generators ξ_i, $i = 1, \ldots, n$. $S(n)$ and $\tilde{H}(n)$ are found with the derivations $D \in W(n)$ that are respectively annihilated by a volume form $\mathcal{V}$, or a (closed) Hamiltonian form $\mathcal{h}$, these forms being defined on a particular superalgebra, $\theta(n)$, built over $\Lambda(n)$ (Kac 1977):

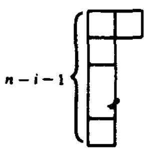
$$S(n) = \{D \in W(n) | D\mathcal{V} = 0\}$$

$$\tilde{H}(n) = \{D \in W(n) | D\mathcal{h} = 0\}.$$

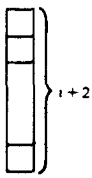
More precisely we have $W(n) = \oplus_{i=-1}^{n-1} W(n)_i$, $S(n) = \oplus_{i=-1}^{n-2} S(n)_i$ and $\tilde{H}(n) = \oplus_{i=-1}^{\Delta-2} \tilde{H}(n)_i$, where

$W(n)_i =$ 

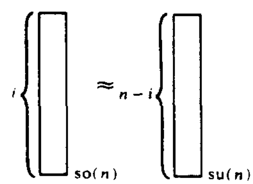
of $\mathrm{u}(n) \approx \mathrm{su}(n) + \mathrm{u}(1)$

$S(n)_i =$ 

of $\mathrm{su}(n)$

$\tilde{H}(n)_i =$ 

of $\mathrm{so}(n)$.

If we note that as a vector space 

, we can identify as vector spaces:

$W(n)$ with $V\left(\overset{1}{\bigcirc}-\overset{0}{\bigcirc}-\cdots-\overset{0}{\bigcirc}-\overset{-n}{\otimes}\right)_{\mathrm{MR}}$,

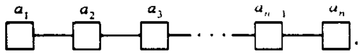
$S(n)$ with $V\left(\overset{1}{\bigcirc}-\overset{0}{\bigcirc}-\cdots-\overset{0}{\bigcirc}-\overset{-n}{\otimes}\right)_{\mathrm{IR}}$, and

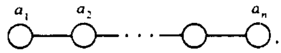
$\tilde{H}(n)$ with $I\left(\overset{1}{\bigcirc}-\overset{0}{\bigcirc}-\cdots-\overset{0}{\bigcirc}-\overset{-n}{\otimes}\right)$.

Therefore we can regard $S(n)$ and $\tilde{H}(n)$ as some irreducible subspaces of the 'non-typical' $W(n)$.

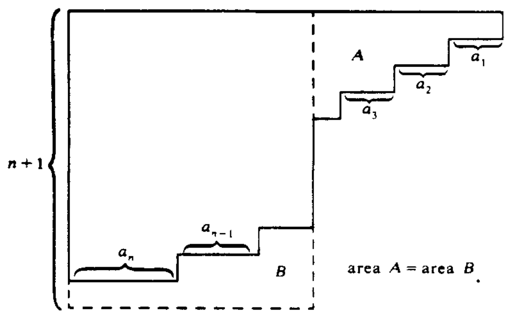
In fact, this is the inverse approach of the non-typicality, where in order to characterise the representations (Farmer and Jarvis 1983) one puts constraints on the non-typical superfields on which the generators realised as differential operators act. Here, the constraints are put on the differential operators.

(viii) In the case of the basic LSA, the YST approach for the superrepresentations was fruitful, mainly in its 'notation' interpretation. This may also be true for some other simple LSA. For example, Leites and Serganova (1984) proposed the following Dynkin diagram for the identification of the $Q(n)$-IR:



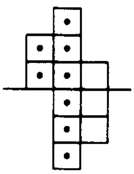
As $Q(n)_0 \approx \mathrm{su}(n+1)$, the representation $V_0(\Lambda)$ is specified by 

If we identify the HW-YST of $V(\Lambda)$ with the YT of $V_0(\Lambda)$, then the YST of any faithful representation is made of more than n boxes. Indeed, in the typical case, all the Dynkin indices are strictly positive, which implies that the YST are made of more than $\Sigma_{i=1}^{n}(n-i)=\frac{1}{2}(n+1)n$ boxes. In the non-typical case, i.e. when at least one of the Dynkin indices vanishes (Kac 1977), the YST is made of a multiple of $n+1$ boxes. The consistency condition $a_i=0$, implying $a_1+2a_2+\ldots+(i-1)a_{i-1}=a_n+2a_{n-1}+\ldots+(n-i)a_{i+1}$, is clearly responsible for that fact, as we can verify with the aid of the following YST:



According to the conclusion of Jarvis and Murray (1985), it is impossible to define $Q(n)$ T-YST in agreement with the standard definitions, and similarly for the $P(n)$ class of LSA.

(ix) Even for the basic LSA, the level of generality of the previously described SRR and their YST is probably not very high: the SRR which appear when making tensor products of IR, as on the RHS of (8.9), can also appear in the reduction of the tensor product of a HW-MR by a LW-MR. Therefore, it could perhaps be advantageous to define YST like this:



for describing the indecomposable subspaces of

which are not themselves (P)HW representations.

If it can be conjectured that some YST, like those above, are needed to describe the SRR built by induction on an indecomposable finite representation of the non-semisimple subsuperalgebra $P = G_0 + G_{+1}$ of G (i.e. the SRR that generalise the su(2/1) SRR made of three or four irreducible subspaces) it is hard to imagine any suitable YST able to describe the representations characterised by a non-diagonal Cartan subalgebra.

Acknowledgments

I am especially grateful to B Morel, A Sciarrino and J Thierry-Mieg for helpful remarks, and to H Ruegg and T Schücker for their encouragement.

Appendix 1

Table A1. Classification of the simple LSA†.

classical LSA (G_0 is reductive)	basic LSA ($\exists B$)	$K \not\equiv 0$:	$A(m, n) m \neq n$, $B(m, n)$
			$C(n)$, $D(m, n) m \neq n+1$
			$D(2, 1, \alpha)$, $G(3)$, $F(4)$
			any simple LA
		$K \equiv 0$:	$A(n, n)$, $D(n+1, n)$
	strange LSA ($\nexists B$):		$P(n)$, $Q(n)$
Cartan-type LSA ($G_{\bar{0}}$ is not reductive; $\nexists B$): $W(n)$, $S(n)$, $\tilde{S}(n)$, $H(n)$			

† In table A1, B denotes any non-degenerate supertrace form and K is the Killing form (other types of bilinear forms are defined in Leites and Serganova (1984)); if K is degenerate we note this by $K \equiv 0$, otherwise $K \not\equiv 0$. The possible range of the parameters m, n, α is specified in tables A6 and A7.

Table A2. Isomorphisms of simple LSA.

$A(m, n) \approx A(n, m)$; $A(0, 1) \approx C(2) \approx W(2)$; $A(1, 1) \approx H(4)$;
$B(0, 1) \approx \tilde{S}(2)$; $D(2, 1, 1) \approx D(2, 1)$; $D(2, 1, \alpha) \approx D(2, 1, \beta)$ if $\beta = \alpha$,
$1/\alpha$, $-(1+\alpha)$, $-1/(1+\alpha)$, $-(1+\alpha)/\alpha$ or $-\alpha/(1+\alpha)$; $\forall \alpha \notin C - \{0, -1, \infty\}$,
(for any $\alpha \in R$, there is an unique $\beta \geqslant 1$); $P(2) \approx S(3)$.

Table A3. Other notations.

Kac (1977, 1978)		Scheunert (1979, 1984, 1985)	Leites and Serganova (1984)	This paper
$A(m, n)$	$\mathrm{sl}(m/n)$	$\mathrm{spl}(m/n)$	$\mathrm{sl}(m/n)$	$\mathrm{su}(m/n)$
$A(n-1, n-1)$	$\mathrm{sl}(n/n)/\lambda 1_{2n}$	$\mathrm{spl}(n/n)/Z_{2n}$	$\mathrm{psl}(n/n)$	$\mathrm{su}(n/n)/\lambda 1_{2n}$
$B(m, n)$	$\mathrm{osp}(2m+1/2\ )$			
$C(n)$	$\mathrm{osp}(2/2n-2)$			$C(1, n-1)$
$D(m, n)$	$\mathrm{osp}(2m/2n)$			$C(m, n)$
$D(2, 1, \alpha)$	$\mathrm{osp}(4/2, \alpha)$	$\Gamma(\sigma_1, \sigma_2, \sigma_3)$	$D(\alpha)$	
$F(4)$		Γ_3	AB_3	
$G(3)$		Γ_2	AG_2	
$P(n)$		$b(n+1)$	$s\pi(n)$	
$Q(n)$		$d(n+1)/Z_{2n+2}$	$\mathrm{psq}(n)$	
$W(n)$			$W(o/n)$	
$S(n)$			$S(o/n)$	
$\tilde{S}(n)$			$S'(n)$	
$H(n)$			$SH(n)$	

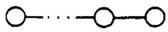
Table A4. The basic Lie superalgebras.

G	G_0	$G_{\bar{0}}$ module G_1	dim G	sdim G
$\mathrm{su}(m/n)$	$\mathrm{su}(m)+\mathrm{su}(n)+\mathrm{u}(1)$	$(m, \bar{n})_1+(\bar{m}, n)_{-1}$	$(m+n)^2-1$	dim $\mathrm{su}(m-n)$
$A(n-1, n-1)$	$\mathrm{su}(n)+\mathrm{su}(n)$	$(n, \bar{n})+(\bar{n}+n)$	$(2n)^2-2$	-2
$\mathrm{osp}(2/n)$	$\mathrm{sp}(n)+u(1)$	n_1+n_{-1}	$\frac{1}{2}(n^2+5n+2)$	dim $\mathrm{so}(2-n)$†
$\mathrm{osp}(m/n)$	$\mathrm{sp}(n)+\mathrm{so}(m)$	(m, n)	$\frac{1}{2}[(m+n)^2+m-n)]$	dim $\mathrm{so}(m-n)$†
$\mathrm{osp}(4/2, \alpha)$	$\mathrm{su}(2)+\mathrm{su}(2)+\mathrm{su}(2)$	$(2, 2, 2)$	17	dim $\mathrm{u}(1)$
$G(3)$	$\mathrm{su}(2)+G_2$	$(2, 7)$	31	dim $\mathrm{su}(2)$
$F(4)$	$\mathrm{su}(2)+\mathrm{so}(7)$	$(2, 8)$	40	dim $\mathrm{su}(3)$

† When $\delta = m-n<0$, then $\dim \mathrm{so}(\delta) = \dim(\text{adjoint}) \equiv \dim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}$, but $\dim \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} = \frac{1}{2}\delta(\delta-1) = \frac{1}{2}(m-n)(m-n-1) = \frac{1}{2}(n-m)(n-m+1)$($\equiv \dim \mathrm{sp}(n-m)$ when $\delta = 2k$). The dimension of the representations of $\mathrm{so}(\delta)$ is formally the same as for $\mathrm{so}(d)$, $d>0$, but now this is δ which has to figure in the 'product of the boxes'. Example: $\dim(\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array})_{\mathrm{IR}} = \frac{1}{2}[\delta(\delta+1)]-1 = \frac{1}{2}(m-n)(m-n+1)-1$.

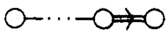
Table A5. Dynkin diagrams and Cartan matrices of simple LA.

su(n):



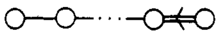
$$A_{n-1}=\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \cdot & & \\ & & \cdot & \cdot & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

so($2n+1$):



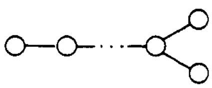
$$B_n=\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \cdot & & \\ & & \cdot & \cdot & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -2 & 2 \end{pmatrix}$$

sp($2n$):



$$C_n=\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & \cdot & & \\ & & \cdot & \cdot & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}$$

so($2n$):



$$D_n=\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & \cdot & & & \\ & \cdot & \cdot & -1 & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix}$$

G_2:

$$G_2=\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Table A6. The infinite series of basic Lie superalgebras.

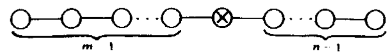
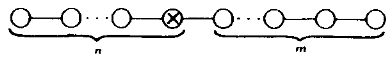
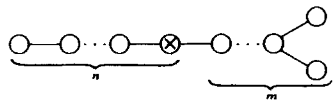
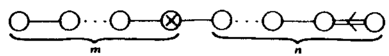
	Roots	Simple roots	Kac–Dynkin diagrams	Cartan matrices
su(m/n) $mn \geqslant 2$	$\Delta_{\bar 0} = \{\varepsilon_i - \varepsilon_j;\ \delta_i - \delta_j\}\, i \neq j$ $\Delta_{\bar 1} = \{\pm(\varepsilon_i - \delta_j)\}\, \forall i, j$	$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $\beta = \alpha_m = \varepsilon_m - \delta_1$ $\gamma_i = \alpha_{m+i} = \delta_i - \delta_{i+1}$		$\left(\begin{array}{c\|c\|c} A_{m-1} & -1 & \\ \hline -1 & 0 & 1 \\ \hline & -1 & A_{n-1} \end{array}\right)$
osp$(2m+1/2n)$, $m \geqslant 1, n \geqslant 1$	$\Delta_{\bar 0} = \{\pm 2\delta_i;\ \pm\delta_i \pm \delta_j;\ \pm\varepsilon_i \pm \varepsilon_j;\ \varepsilon_i\}\, i \neq j$ $\Delta_{\bar 1} = \{\pm\delta_i;\ \pm\varepsilon_i \pm \delta_j\}\, \forall i, j$	$\alpha_i = \delta_i - \delta_{i+1};\ \gamma_i = \varepsilon_i - \varepsilon_{i+1} = \alpha_{n+i}$ $\beta = \alpha_n = \delta_n - \varepsilon_1;\ \gamma_m = \varepsilon_m = \alpha_{m+n}$		$\left(\begin{array}{c\|c\|c} A_{n-1} & -1 & \\ \hline -1 & 0 & 1 \\ \hline & -1 & B_m \end{array}\right)$
osp$(1/2n)$, $n \geqslant 1$	$\Delta_0 = \{\pm 2\delta_i;\ \pm\delta_i \pm \delta_j\}\, i \neq j$ $\Delta_{\bar 1} = \{\pm\delta_i\}\, \forall i$	$\alpha_i = \delta_i - \delta_{i+1}$ $\beta = \alpha_n = \delta_n$		(B_n)
osp$(2/2n-2)$, $n \geqslant 2$	$\Delta_{\bar 0} = \{\pm 2\delta_i;\ \pm\delta_i \pm \delta_j\}\, i \neq j$ $\Delta_{\bar 1} = \{\pm\varepsilon \pm \delta_i\}\, \forall i$	$\gamma_i = \delta_i - \delta_{i+1}$ $\gamma_{n-1} = 2\delta_{n-1}$ $\beta = \gamma_0 = \varepsilon - \delta_1$		$\left(\begin{array}{c\|c} 0 & 1 \\ \hline -1 & \\ & C_{n-1} \end{array}\right)$
osp$(2m/2n)$, $m \geqslant 2, n \geqslant 1$	$\Delta_{\bar 0} = \{\pm 2\delta_i;\ \pm\delta_i \pm \delta_j;\ \pm\varepsilon_i \pm \varepsilon_j\}\, i \neq j$ $\Delta_{\bar 1} = \{\pm\varepsilon_i \pm \delta_j\}\, \forall i, j$	(i) $\alpha_i = \delta_i - \delta_{i+1};\ \gamma_i = \varepsilon_i - \varepsilon_{i+1} = \alpha_{n+i}$ $\beta = \alpha_n = \delta_n - \varepsilon_1;\ \gamma_m = \varepsilon_{m-1} + \varepsilon_m = \alpha_{m+n}$		$\left(\begin{array}{c\|c\|c} A_{n-1} & -1 & \\ \hline -1 & 0 & 1 \\ \hline & -1 & D_m \end{array}\right)$
		(ii) $\gamma_i = \varepsilon_i - \varepsilon_{i+1};\ \alpha_i = \delta_i - \delta_{i+1}$ $\beta = \gamma_m = \varepsilon_m - \delta_1;\ \alpha_n = \delta_{n-1} + \delta_n$		$\left(\begin{array}{c\|c\|c} A_{m-1} & -1 & \\ \hline -1 & 0 & 1 \\ \hline & -1 & C_n \end{array}\right)$

Table A7. The exceptional Lie superalgebras.

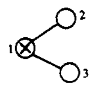
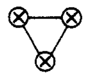
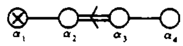
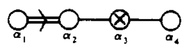
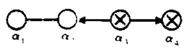
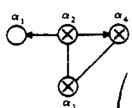
$D(2,1,\alpha)$ $\alpha \neq$ $\{0,-1,\infty\}$	$\Delta_{\bar{0}}=\{\pm 2\varepsilon_i\}$ $\Delta_{\bar{1}}=\{\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\}$	System 1, 2 or 3 $\alpha_1=\varepsilon_1+\varepsilon_2+\varepsilon_3$ $\alpha_2=-2\varepsilon_i$ $\alpha_3=-2\varepsilon_j,\ i\neq j$	 $\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$
		System 4 $\alpha_1=\varepsilon_1+\varepsilon_2+\varepsilon_3$ $\alpha_2=\varepsilon_1-\varepsilon_2-\varepsilon_3$ $\alpha_3=-\varepsilon_1-\varepsilon_2+\varepsilon_3$	
$G(3)$	$\Delta_{\bar{0}}=\{\varepsilon_i-\varepsilon_j;\ \pm\varepsilon_i;\ \pm 2\delta\}$ $\Delta_{\bar{1}}=\{\pm\varepsilon_i\pm\delta;\ \pm\delta\}$	$\alpha_1=\delta+\varepsilon_1$ $\alpha_2=\varepsilon_2$ $\alpha_3=\varepsilon_3-\varepsilon_2$	 $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}$
$F(4)$	$\Delta_{\bar{0}}=\{\pm\varepsilon_i\pm\varepsilon_j;\ \pm\varepsilon_i;\ \pm 2\delta\}_{i\neq j}$ $\Delta_{\bar{1}}=\{\frac{1}{2}(\pm\varepsilon_i\pm\varepsilon_2\pm\varepsilon_3\pm\delta)\}$	$\alpha_1=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta)$ $\alpha_2=-\varepsilon_1$ $\alpha_3=\varepsilon_1-\varepsilon_2$ $\alpha_4=\varepsilon_2-\varepsilon_3$	 $\begin{pmatrix} 0 & 1 & & \\ -1 & 2 & -2 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$
		$\alpha_1=\varepsilon_1-\varepsilon_2$ $\alpha_2=-\varepsilon_1$ $\alpha_3=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta)$ $\alpha_4=-\delta$	 $\begin{pmatrix} 2 & -1 & & \\ -2 & 2 & -1 & \\ & 1 & 0 & -1 \\ & & 1 & 2 \end{pmatrix}$
		$\alpha_1=\varepsilon_3-\varepsilon_2$ $\alpha_2=\varepsilon_2-\varepsilon_1$ $\alpha_3=\frac{1}{2}(\varepsilon_1-\varepsilon_2-\varepsilon_3-\delta)$ $\alpha_4=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta)$	 $\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & 1 & 0 & 1 \\ & & -1 & 0 \end{pmatrix}$
		$\alpha_1=\varepsilon_1-\varepsilon_2$ $\alpha_2=\frac{1}{2}(-\varepsilon_1+\varepsilon_2+\varepsilon_3-\delta)$ $\alpha_3=\frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3+\delta)$ $\alpha_4=\frac{1}{2}(-\varepsilon_1-\varepsilon_2-\varepsilon_3+\delta)$	 $\begin{pmatrix} 2 & -1 & & \\ 1 & 0 & 2 & 1 \\ & -2 & 0 & -3 \\ & -1 & 3 & 0 \end{pmatrix}$

Table A8. The Q, C and H_δ generators, ($h_i \equiv H_{\alpha_i}$).

su(m/n)	$Q = \frac{1}{n-m}\left(n \sum_{i=1}^{m} ih_i - m \sum_{i=1}^{n-1} ih_{m+n-i}\right)$	
su(n/n)	$C = \sum_{i=1}^{n} ih_i - \sum_{i=1}^{n-1} ih_{2n-i}$	
osp($2/2n$)	$Q = h_1 - \sum_{i=1}^{n+1} h_i$	
osp($1/2n$)	$H_{2\varepsilon_n} = \frac{1}{2}h_n$	
osp($2m+1/2n$)	$H_{\varepsilon_{n-1}+\varepsilon_n} = h_n - \sum_{i=1}^{m-1} h_i - \frac{1}{2}h_{n+m}$	
osp($2m/2n$) $-D(m, n)$	$H_{\varepsilon_{n-1}+\varepsilon_n} = h_n - \sum_{i=1}^{m-2} h_i - \frac{1}{2}(h_{m+n-1} + h_{m+n})$	
osp($2m/2n$) $-C(m, n)$	$H_{2\delta_n} = h_{m-1} + 2\left(h_m - \sum_{i=1}^{n} h_{m+i}\right)$	
$D(2, 1, \alpha)$	$H_{2\varepsilon_k} = \frac{1}{1+\alpha}(2h_1 - h_2 - \alpha h_3)$	$k \neq i, j$
$F(4)$	$H_{2\delta} = \frac{1}{3}(2h_1 - 3h_2 - 4h_3 - 2h_4)$	
$G(3)$	$H_{2\delta} = \frac{1}{2}(h_1 - 2h_2 - 3h_3)$	

Table A9. The consistency conditions

let G	if	then, necessarily
$B(m, n)$	$b \leq m-1$	$a_{n+b+1} = \cdots = a_{n+m} = 0$
$D(m, n)$	$b < m-1$	$a_{n+b+1} = \cdots = a_{n+m} = 0$
	$b = m-1$	$a_{n+m-1} = a_{n+m}$
$D(2, 1, \alpha)$	$b = 0$	$a_2 = a_3 = 0$
	$b = 1$	$a_2 + 1 = \alpha(a_3 + 1)$
$F(4)$	$b = 0$	$a_2 = a_3 = a_4 = 0$
	$b = 1$	impossible
	$b = 2$	$a_2 = a_4 = 0$
	$b = 3$	$a_2 = 2a_4 + 1$
$G(3)$	$b = 0$	$a_2 = a_3 = 0$
	$b = 1$	impossible
	$b = 2$	$a_2 = 0$

Table A10. The positive odd roots (if not otherwise specified, the range of the indices are the ones in the brackets).

su$(m/n)(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$
$\varepsilon_i - \delta_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_m + \ldots + \alpha_{m+j-2} + \alpha_{m+j-1}$

$B(m, n)(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m > 0)$
$\delta_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n + \ldots + \alpha_{n+j-2} + \alpha_{n+j-1}$
$\delta_i = \alpha_i + \ldots + \alpha_n + \ldots + \alpha_{n+m-1} + \alpha_{n+m}$
$\delta_i + \varepsilon_j = \alpha_i + \ldots + \alpha_n + \ldots + \alpha_{n+j-1} + 2(\alpha_{n+j} + \ldots + \alpha_{n+m-1}) + \alpha_{n+m}$

$C(n)(1 \leqslant j \leqslant n-1)$
$\varepsilon - \delta_j = \alpha_1 + \alpha_2 + \ldots + \alpha_{j-1} + \alpha_j$
$\varepsilon + \delta_{n-1} = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} + \alpha_n$
$\varepsilon + \delta_j = \alpha_1 + \ldots + \alpha_{j-1} + 2(\alpha_j + \ldots + \alpha_{n+m-1}) + \alpha_{n+m}, j \leqslant n-2$

$D(m, n)(1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$
$\delta_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n + \ldots + \alpha_{n+j-2} + \alpha_{n+j-1}$
$\delta_i + \varepsilon_m = \alpha_i + \ldots + \alpha_n + \ldots + \alpha_{n+m-2} + \alpha_{n+m}$
$\delta_i + \varepsilon_{m-1} = \alpha_i + \ldots + \alpha_n + \ldots + \alpha_{n+m-1} + \alpha_{n+m}$
$\delta_i + \varepsilon_j = \alpha_i + \ldots + \ldots + \alpha_{n+j-1} + 2(\alpha_{n+j} + \ldots + \alpha_{n+m-2}) + \alpha_{n+m-1} + \alpha_{n+m}, j \leqslant m-2$

$D(2, 1, \alpha)(p, q = \pm 1)$
$\varepsilon_1 + p\varepsilon_2 + q\varepsilon_3 = \alpha_1 + m\alpha_2 + n\alpha_3$, where $m = \frac{1}{2}(1-p)$, $n = \frac{1}{2}(1-q)$

$F(4)(p, q, r = \pm 1)$
$\delta + p\varepsilon_1 + q\varepsilon_2 + r\varepsilon_3 = \alpha_1 + k\alpha_2 + m\alpha_3 + n\alpha_4$,
where $k = \frac{1}{2}(3-p-q-r)$, $m = \frac{1}{2}(2-q-r)$, $n = \frac{1}{2}(1-r)$

$G(3)$ (the seven positive roots are δ; $\delta \pm \varepsilon_i$, $i = 1, 2, 3$)
$\delta + p\varepsilon_1 + q\varepsilon_2 + r\varepsilon_3 = \alpha_1 + m\alpha_2 + n\alpha_3$, where $m = 2-2p+q+r$ and $n = 1-p+r$
(remember that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$)

Table A11. The non-typicality conditions.

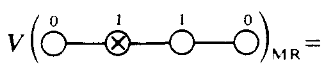
let G	and α	then $(\Lambda + \rho, \alpha) = 0$ if
su(m/n)	$\sum_{t=i}^{j} \alpha_t$	$\sum_{t=i}^{m-1} (a_t + 1) + a_m - \sum_{t=m+1}^{j} (a_t + 1) = 0$
$B(m, n)$	$\sum_{t=1}^{n+m} n_t\alpha_t$	$\sum_{t=1}^{n-1} n_t(a_t + 1) + a_n - \sum_{t=n+1}^{n+m} n_t(a_t + 1) = 0$
$C(n)$	$\sum_{t=1}^{n} n_t\alpha_t$	$a_1 - \sum_{t=2}^{n-1} n_t(a_t + 1) - 2n_n(a_n + 1) = 0$
$D(m/n)$	$\sum_{t=1}^{n+m} n_t\alpha_t$	$\sum_{t=1}^{n-1} n_t(a+1) + a_n - \sum_{t=n+1}^{n+m-1} n_t(a+1) - \frac{1}{2}n_{n+m}(a_{n+m} + 1) = 0$
$D(2, 1, \alpha)$	$\alpha_1 + m\alpha_2 + n\alpha_3$	$a_1 - m(a_2 + 1) - \alpha n(a_3 + 1) = 0$
$F(4)$	$\alpha_1 + k\alpha_2 + m\alpha_3 + n\alpha_4$	$a_1 - k(a_2 + 1) - 2m(a_3 + 1) - 2n(a_4 + 1) = 0$
$G(3)$	$\alpha_1 + m\alpha_2 + n\alpha_3$	$a_1 - m(a_2 + 1) - 3n(a_3 + 1) = 0$

Table A12. Type-2 LSA non-typicality conditions (other formulation).

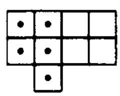
$B(m, n)$	$\delta_i - \varepsilon_j$	$\sum_{l=i}^{n-1} a_l + b + \sum_{l=n+j}^{n+m-1} a_l + \frac{1}{2}a_{n+m} + (n+1-i-j) = 0$
	δ_i	$\sum_{l=i}^{n-1} a_l + b - (n-m-i+\frac{1}{2}) = 0$
	$\delta_i + \varepsilon_j$	$\sum_{l=i}^{n-1} a_l + b - \sum_{l=n+j}^{n+m-1} a_l - \frac{1}{2}a_{n+m} + (n-2m-i-j) = 0$
$D(m, n)$	$\delta_i - \varepsilon_j$	$\sum_{l=i}^{n-1} a_l + b + \sum_{l=n+j}^{n+m-2} a_l + \frac{1}{2}(a_{n+m-1} + a_{n+m}) + (n+1-i-j) = 0$
	$\delta_i - \varepsilon_m$	$\sum_{l=i}^{n-1} a_l + b \mp \frac{1}{2}(-a_{n+m-1} + a_{n+m}) + (n+1-m-i) = 0$
	$\delta_i + \varepsilon_m$	$\sum_{l=i}^{n-1} a_l + b + \frac{1}{2}(a_{n+m-1} - a_{n+m}) + (n+1-m-i) = 0$
	$\delta_i + \varepsilon_j$	$\sum_{l=i}^{n-1} a_l + b - \sum_{l=n+j}^{n+m-2} a_l - \frac{1}{2}(a_{n+m-1} + a_{n+m}) + (n-2m+1-i+j) = 0$
$D(2, 1, \alpha)$	$\varepsilon_1 + p\varepsilon_2 + q\varepsilon_3$	$(b + pa_2 + p - 1) + \alpha(b + qa_3 + q - 1) = 0$
$F(4)$	$\delta + p\varepsilon_1 + q\varepsilon_2 + r\varepsilon_3$	$3b + (p+q+r)a_2 + 2(q+r)a_3 + 2ra_4 + (p+3q+5r) - 9 = 0$
$G(3)$	$\delta + p\varepsilon_1 + q\varepsilon_2 + r\varepsilon_3$	$2b + (2p-q-r)a_2 + 3(p-r)a_3 + (5p+q+4r) - 5 = 0$

Appendix 2

Example 1. Consider

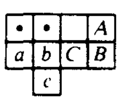


$(1,3)_2 + (2, 6+3^*)_1 + (3, 8+1)_0 + (1, 10+8)_0 + (2, 15+6^*+3)_{-1} + (4,3)_{-1} + \ldots + (1,3)_{-4}$

which can be described by the following YST: 

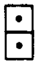
In particular $b_1 + (2-1) = \bar{b}_2 + (3-2)$ is the only non-typicality condition. Therefore there is only one invariant subspace.

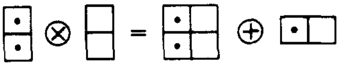
According to the third method, the YST characterising $I(\Lambda)$ can be easily determined.

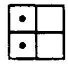
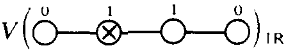
Starting from the YST 

of $V(\Lambda)_{MR}$, we have to remove successively

the boxes	due to
A and a	i
B	ii
b	iii
c	ii
C	iii

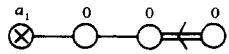
Thus, according to the third method, $I(\Lambda)$ is specified by , i.e. $I(\Lambda)_0 = (2, 6^*)_{-1}$. This can be independently checked by considering the tensor product of  $= (1, 1)_6 + (2, 3)_5 + (1, 6)_4$ by  $= (1, 3)_{-4} + (2, 3^*)_{-5} + (3, 1)_{-6}$. Indeed,



where  denotes the adjoint $(2, 3^*)_1 + (3+1, 1)_0 + (1, 8)_0 + (2, 3)_{-1}$, and  denotes  $=$

$$[(1, 3)_2 + (2, 6+3^*)_1 + (3, 8+1)_0 + (1, 10+8)_0 + (2, 15+3)_{-1} + (4, 3)_{-1} + (3, 6)_{-2}].$$

By comparison with $V(\Lambda)_{MR}$, we see that $I(\Lambda)_0 = (2, 6^*)_{-1}$.

Example 2. Consider the following class of MR: . Then, according to § 2,

$$V(\Lambda)_{MR} = 1_q + 6_{q-1} + (14+1)_{q-2} + (14'+6)_{q-3} + (14+1)_{q-4} + 6_{q-5} + 1_{q-6}$$

of $sp(6) + u(1)$, where $q = a_1$.

In the non-typical case, with the aid of the second method, we find

$a_1 = 0$ $\quad V(\Lambda)_{IR} = 1_0$

$\quad I(\Lambda) = 6_{-1} + (14+1)_{-2} + (14'+6)_{-3} + (14+1)_{-4} + 6_{-5} + 1_{-6}$

$a_1 = 1$ $\quad V(\Lambda)_{IR} = 1_1 + 6_0 + 1_{-1}$

$\quad I(\Lambda) = 14_{-1} + (14'+6)_{-2} + (14+1)_{-3} + 6_{-4} + 1_{-5}$

$a_1 = 2$ $\quad V(\Lambda)_{IR} = 1_2 + 6_1 + (14+1)_0 + 6_{-1} + 1_{-2}$

$\quad I(\Lambda) = 14'_{-1} + 14_{-2} + 6_{-3} + 1_{-4}$

$a_1 = 4$ $\quad V(\Lambda)_{IR} = 1_4 + 6_3 + 14_2 + 14'_1$

$\quad I(\Lambda) = 1_2 + 6_1 + (14+1)_0 + 6_{-1} + 1_{-2}$

$a_1 = 5$ $\quad V(\Lambda)_{IR} = 1_5 + 6_4 + (14+1)_3 + (14'+6)_2 + 14_1$

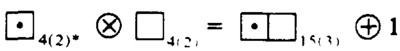
$\quad I(\Lambda) = 1_1 + 6_0 + 1_{-1}$

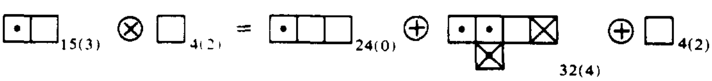
$a_1 = 6$ $\quad V(\Lambda)_{IR} = 1_6 + 6_5 + (14+1)_4 + (14'+6)_3 + (14+1)_2 + 6_1$

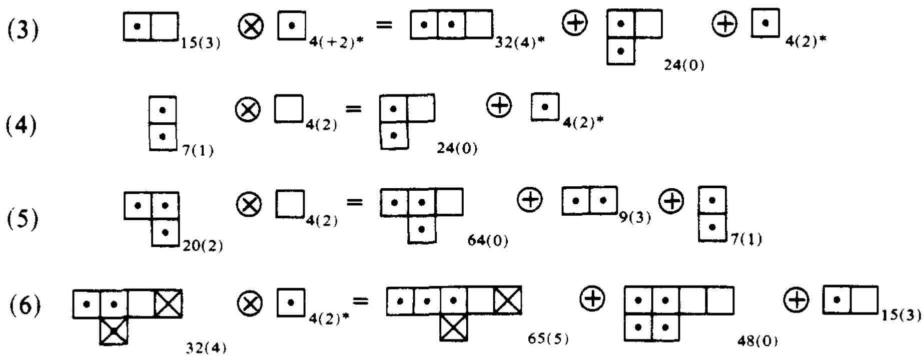
$\quad I(\Lambda) = 1_0$.

Appendix 3

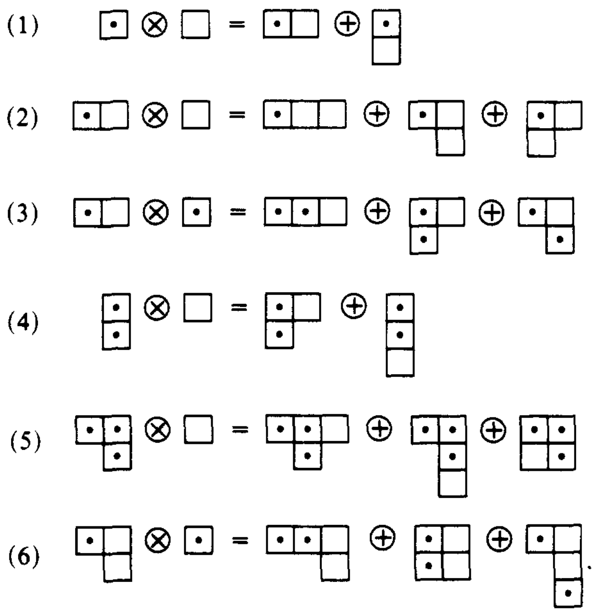
The following su(1/3) tensor products, expressed in terms of HW-YST,

(1) 

(2) 



appear to look simpler when expressed in terms of T-YST:



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